# A TEXTBOOK OF
# MATHEMATICS
## FOR CLASSES XI-XII
## BOOK IV

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# A TEXTBOOK OF MATHEMATICS

## FOR CLASSES XI-XII

## BOOK IV

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# FOREWORD

This is the fourth and final book in the series "A Textbook of Mathematics for Classes XI-XII". Definite lines for its development and presentation of the subject matter were evolved by the mathematics group in the Department of Education in Science and Mathematics (DESM) at the National Council of Educational Research and Training (NCERT) under the guidance of Prof. Manmohan Singh Arora. Draft materials were prepared by a group of teachers, subject specialists and users of Mathematics, keeping as central the guidelines developed, at a workshop held specifically for this purpose at the Department of Mathematics, Kurukshetra University, Kurukshetra, in October 1977. I record my appreciation of the important contribution which each one of the participants has made in shaping this book.

Final writing and content editing was done by Prof. Manmohan Singh Arora. He received valuable assistance from Dr. Ajit Kaur Chilana, Department of Mathematics, University of Delhi; Dr. K. P. Chinda, Department of Mathematics, Hindu College, Delhi; Dr. Satish Shirali, Centre for Advanced Studies in Mathematics, Panjab University and Dr. R. P. Gupta of DESM.

Mention must also be made of the valuable contribution of Dr. Ram Autar, Dr. S. K. Singh Gautam and Dr. K. C. Madan of DESM who assisted the editorial team in many ways. The Hindi version was looked after by Dr. K. C. Madan and Shri Mahendra Shanker. Answers were provided by Dr. S. C. Das and Shri G. D. Dhall.

I am highly appreciative of the long hours each one has put in to have this book ready well in time for the second semester. I record my gratitude to each one of them, particularly, to Prof. Manmohan Singh Arora who carried the brunt of the burden in editing this book in addition to his other duties.

The Council will be grateful to the users of this book for their reactions and any suggestions for further improvement in its next edition.

SHIB K. MITRA
*Director*
National Council of Educational
Research and Training

New Delhi
*October 1978*

# PREFACE

This book is fourth in the series "A Textbook of Mathematics for Classes XI-XII." On the cardinal question, 'Should there be separate mathematics courses for different categories of users, viz., students of biology, physics, economics, etc., or only one core course to meet the needs of all students for Classes XI-XII ?', we contacted several experienced teachers and subject specialists. They all favoured **one core course in mathematics at this stage with situations drawn from various fields of application.**

With a detailed outline developed at the Council, taking full cognizance of the prerequisites of the child, his expected level of attaintment, capability for intake and several allied factors, the first draft of the book was prepared in the Department of Education in Science and Mathematics (DESM) This was taken to a workshop organised for the purpose at the Department of Mathematics, Kurukshetra University, Kurukshetra, to which several teachers and subject experts were invited, for its critical review. I am very grateful for the contribution each one of the participants of the above workshop made as also for the work done by my colleagues at the Council.

I then undertook to edit and rewrite the materials with the valued assistance of Dr. Ajit Kaur Chilana, Department of Mathematics, University of Delhi; Dr. K. P. Chinda, Department of Mathematics, Hindu College, Delhi; Dr. Satish Shirali, Centre for Advanced Studies in Mathematics, Panjab University, Chandigarh; and Dr. R. P. Gupta of DESM. Dr. Ram Autar, Dr. S. K. Singh Gautam and Dr. K. C. Madan of DESM provided valuable support in more ways than one. The Hindi version was looked after by Dr. K. C Madan and Shri Mahendra Shanker. Dr. S. C. Das and Shri G. D. Dhall provided the answers. I record my deep gratitude to each one of them for their unstinted co-operation and assistance without which it would not have been possible to edit this book in so short a time.

It is in the fitness of things to draw attention to some of the salient features of this book which are mentioned below:

1. The book keeps to the same style as the Council's books for classes IX and X. Understanding of concepts and results has been emphasized, often by appealing to the heuristics and even at the cost of rigour at times.

2. Suitable examples and situational motivation have been provided while introducing a concept.

3. There is an abundance of solved examples to let the student appreciate the wide area of application of a concept.

4. The book is divided into self-contained units. Each unit opens with a preview of what is contained in it and is followed by a list of key concepts to enable the child to quickly review and recall what he has studied in a particular unit.

5. Suggestions for further reading are provided at the end of every unit to let the interested students expand their horizons.

6. The numerous exercises in each unit are intended to place in the hands of the teacher a stock of questions in order that he can meet adequately the needs of the students *with differing ability levels.*

7. Recall sections, units and exercises have been inserted to highlight the concepts that a child ought to grasp before he goes on to the next section or next unit.

8. Miscellaneous exercises have been included at suitable places so that the student can review the materials at appropriate intervals and attain proficiency in the subject.

9. 'Why ?' has been added at several places to whet the student's mathematical thinking and to help him to crystallise his ideas while searching for the answers. The teacher is, therefore, expected to prod the student to supply the answers rather than simply give the answers himself.

10. Harder problems and sections are marked with asterisks.

11. Historical anecdotes have been introduced to let the student peep into the process of evolution of a particular concept and see what goes on in the mind of a keen student of mathematics.

Here again I must say a **few words to the student.**

*You are about to study this book in mathematics. If you imbibe certain 'good' learning habits from the outset, you will find the study of mathematics captivating and rewarding. Some of the 'good' learning habits are mentioned below :*

1. *Mathematics is best learnt by doing only. Do not just read your textbook. You should always have a pencil and paper with you and 'work through' the text.*

2. *At places where you come across 'Why ?' in the text, you should strive to discover the answer.*

3. *Never spend too much time on a particular problem. It is always better to go on to the next and revert to the one that is giving trouble a little later, with a fresh mind.*
4. *The human brain is capable of storing only a limited amount of information. What is not used frequently is eliminated from the storage. You will do well, therefore, to make summaries of basic results in every unit and frequently review them.*

Despite our best efforts, it is possible that some unintentional errors might have eluded us. The Council will acknowledge with much gratitude when any of these are pointed out and will also welcome the suggestions of its users for improvement in the future editions of this book.

MANMOHAN SINGH ARORA
*Editor*

New Delhi
October 1978

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The National Council of Educational Research and Training is grateful to the following who made a significant contribution in reviewing a draft prepared for this textbook in a Workshop held for the purpose at the Department of Mathematics, Kurukshetra University, Kurukshetra in October, 1977 :

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# UNIT XXIV

# DERIVATIVES AND GRAPHS

*In this unit, we use our understanding of the geometrical interpretation of the derivative to draw graphs of functions. We also introduce the notion of continuity of a function.*

## 24.1 Introduction

So far we have drawn graphs of functions by plotting some points and joining them by free hand curves. We recall, for instance, how we draw the graph of $y=x^2$. We construct a table of values of $x$ and $y$, usually taking enough pairs of values so that when we plot them, we get a '*rough*' idea of what the graph looks like. We then join these points by a free hand curve. *But what gives us the right to join these points by a free hand curve? How do we know that the graph, for instance, does not undulate (have waves) between two consecutive points as shown in Fig. 24.1?*

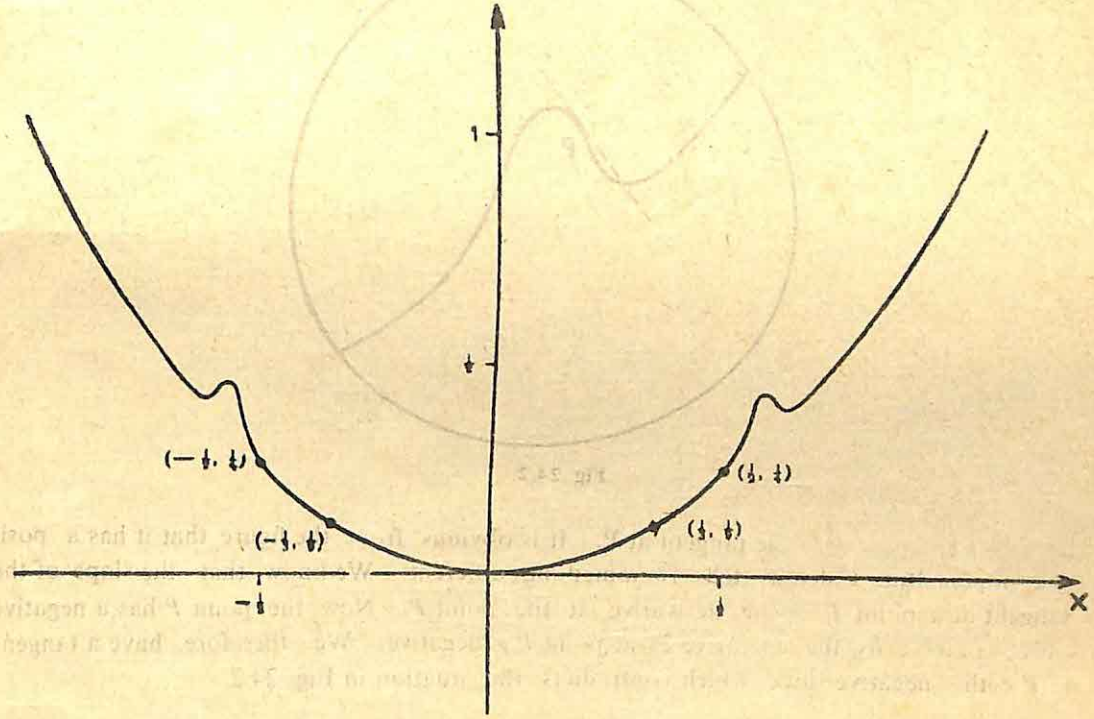

**Fig. 24.1** : Can this be the graph of $y=x^2$ ?

The answer to these questions is that so far we do not really have any justification for joining the points by a free hand curve and that we do not really know whether or not the graph undulates between two consecutive points. It is for this reason that we take *close enough* points and hope not to commit any serious errors.

We will now see how the geometrical meaning of derivative can be used to draw *better* graphs.

Before, however, we do so, let us use our knowledge of Calculus (in particular, the derivative) to see why graph of $y=x^2$ cannot really undulate (have waves). Let us magnify the graph of $y=x^2$ in a neighbourhood of a point, say, $\left(-\frac{1}{2}, \frac{1}{4}\right)$. (See Fig. 24.2) Since $y=x^2$ has a derivative (and hence a tangent) for every value of $x$, there must be a tangent at the point P as shown in the figure.

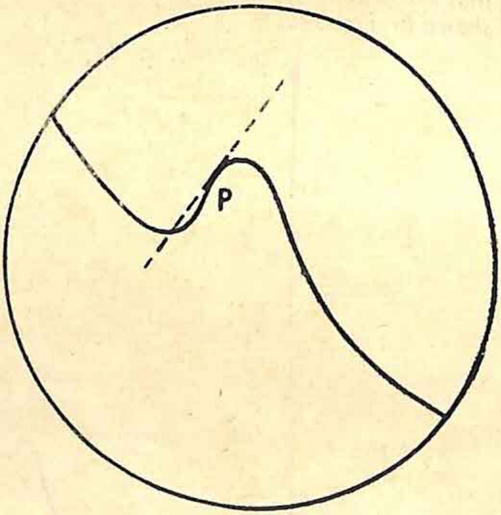

Fig. 24.2

Let us examine the tangent at P. It is obvious from the figure that it has a positive slope. But Calculus tells us something different. We know that the slope of the tangent at a point $P$ is the derivative at the point $P$. Now, the point $P$ has a negative abscissa and, thus, the derivative $2x$ at point $P$ is negative. We, therefore, have a tangent at $P$ with a negative slope, which contradicts the situation in Fig. 24.2.

This is why the graph of $y=x^2$ cannot have waves between two consecutive points.

We will now learn how to use this knowledge of Calculus systematically in applications to graphing.

## 24.2 Increasing and Decreasing Functions

Let us consider the function $f(x)=2x+3$. We recall that its graph is a line. (See Fig. 24.3)

Let us take any two values of $x$ and let us call the smaller one $x_1$ and the larger one $x_2$. We observe* from the graph that $f(x_1)<f(x_2)$.

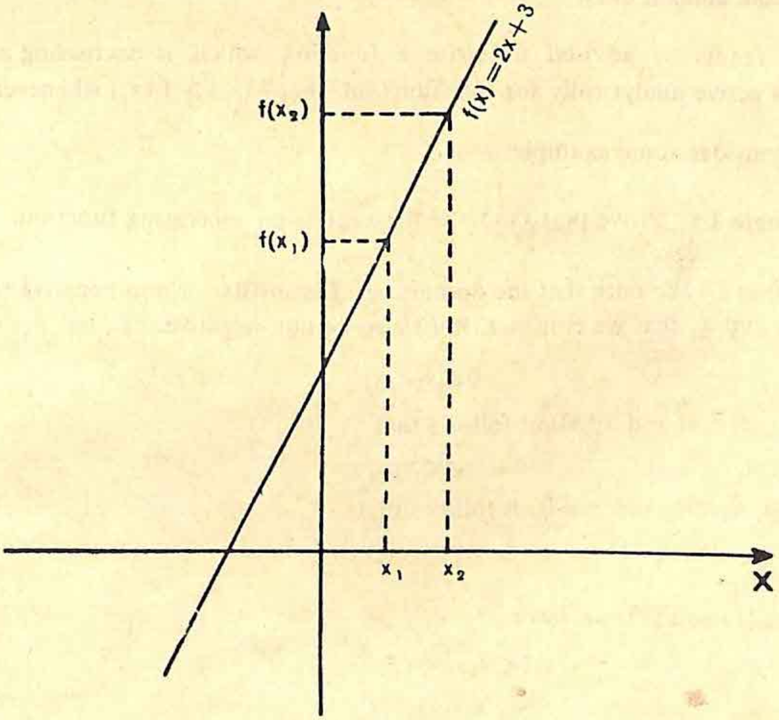

Fig. 24.3

There are many other functions which have this property, namely that, if $x_1<x_2$, then $f(x_1)<f(x_2)$. In other words, *the function values increase as x increases.*

---

[*] This can, in fact, be proved.
Since $x_1<x_2$, it follows that $2x_1<2x_2$ and hence $2x_1+3<2x_2+3$ which means that $f(x_1)<f(x_2)$.

*We say,*

**A function is increasing* if $f(x_1)<f(x_2)$ whenever $x_1<x_2$ (obviously for all $x_1$ and $x_2$ in the domain of f).**

*If, however, the function values decrease as x increases, we have what we call a* **decreasing function.**

We say,

**A function is decreasing** if $f(x_1)>f(x_2)$ whenever $x_1<x_2$ (obviously for all $x_1$ and $x_2$ in the domain of f).**

[The reader is advised to write a function which is decreasing and sketch its graph. Then prove analytically for this function that $f(x_1)>f(x_2)$ whenever $x_1<x_2$.]

We consider some examples :

**Example 1 :** Prove that $f(x)=x^2$ for $x\geq 0$ is an increasing function.

**Solution :** We note that the domain of $f$ consists of non-negative real numbers. Therefore, $x_1$ and $x_2$ that we consider, must also be non-negative, *i.e.*, let

$$0\leqslant x_1<x_2$$

Since, $x_1<x_2$ and $x_1\geqslant 0$, it follows that

$$x_1^2\leqslant x_1x_2 \tag{1}$$

Again, $x_1<x_2$ and $x_2>0$, it follows that

$$x_1x_2<x_2^2 \tag{2}$$

From (1) and (2), we have

$$x_1^2\leqslant x_1x_2<x_2^2$$

*i.e.*,

$$x_1^2<x_2^2$$

which means that

$$f(x_1)<f(x_2)$$

Hence, $f$ is increasing.

---

* Some authors call such a function *strictly increasing*.

** Some authors call such a function *strictly decreasing*.

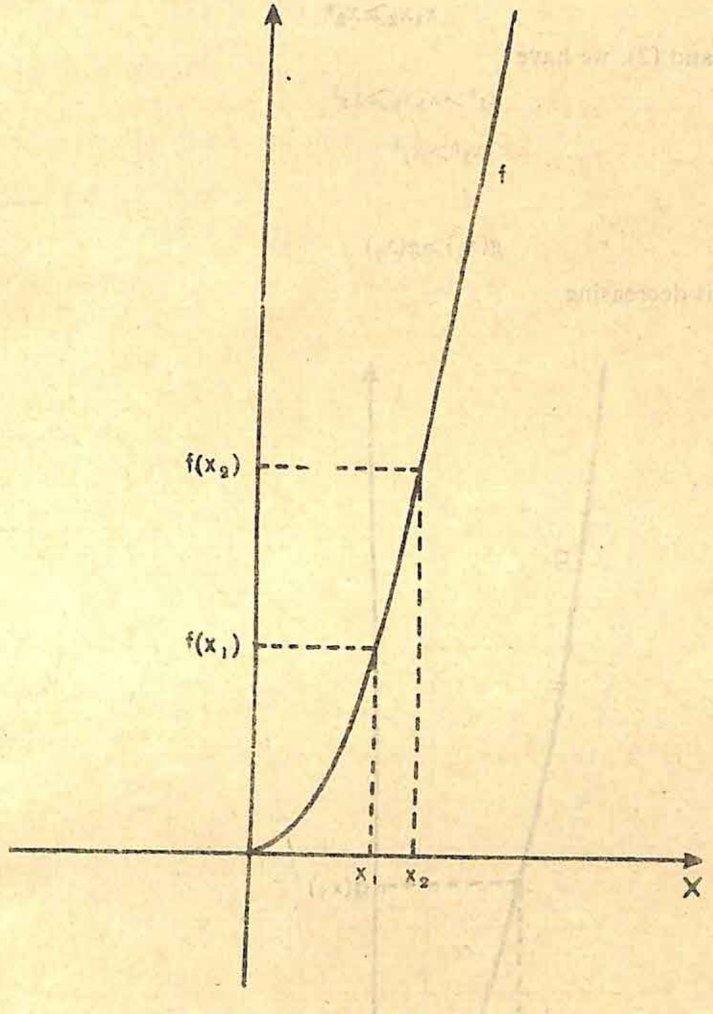

Fig. 24.4 : Graph of $f(x)=x^2,\ x \geqslant 0$

**Example 2 :** Prove that $g(x)=x^2$ for $x \leqslant 0$ is a decreasing function.

**Solution :** Let us take $x_1$ and $x_2$ in the domain of *f*, *i.e.*, let

$$x_1 < x_2 \leqslant 0$$

Since, $x_1 < x_2$ and $x_1 < 0$, it follows that

$$x_1^2 > x_1 x_2 \qquad (1)$$

Again, $x_1 < x_2$ and $x_2 \leqslant 0$, it follows that

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$$x_1x_2 \geqslant x_2^2 \qquad (2)$$

From (1) and (2), we have

$$x_1^2 > x_1x_2 \geqslant x_2^2$$

*i.e.*,

$$x_1^2 > x_2^2$$

which means that

$$g(x_1) > g(x_2)$$

Hence, $g$ is decreasing.

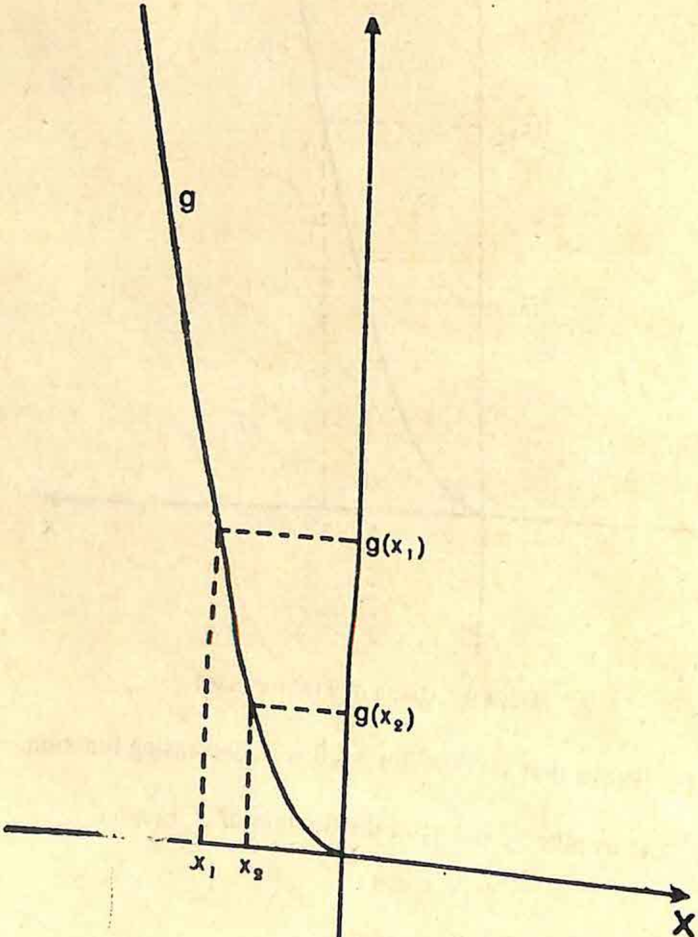


Fig. 24.5 : Graph of $g(x)=x^2$, $x \leqslant 0$

**Example 3 :** Prove that $\phi(x)=x^2$ for all real $x$ is **neither increasing nor decreasing.**

**Solution :** Let us take two values of $x$, say, $x_1=-2$ and $x_2=3$. We have $\phi(x_1)=4$ and $\phi(x_2)=9$. We see that $x_1<x_2$ and $\phi(x_1)<\phi(x_2)$. We are, therefore, *tempted to say that $\phi$ is increasing*. But is the above true for all $x_1<x_2$ ?

Let $x_1=-4$ and $x_2=1$. Then $\phi(x_1)=16$ and $\phi(x_2)=1$. Here is, therefore, a pair of values of $x_1$ and $x_2$ for which $x_1<x_2$ but $\phi(x_1)$ is no longer less than $\phi(x_2)$.

**We, thus, conclude that $\phi$ is not increasing.**

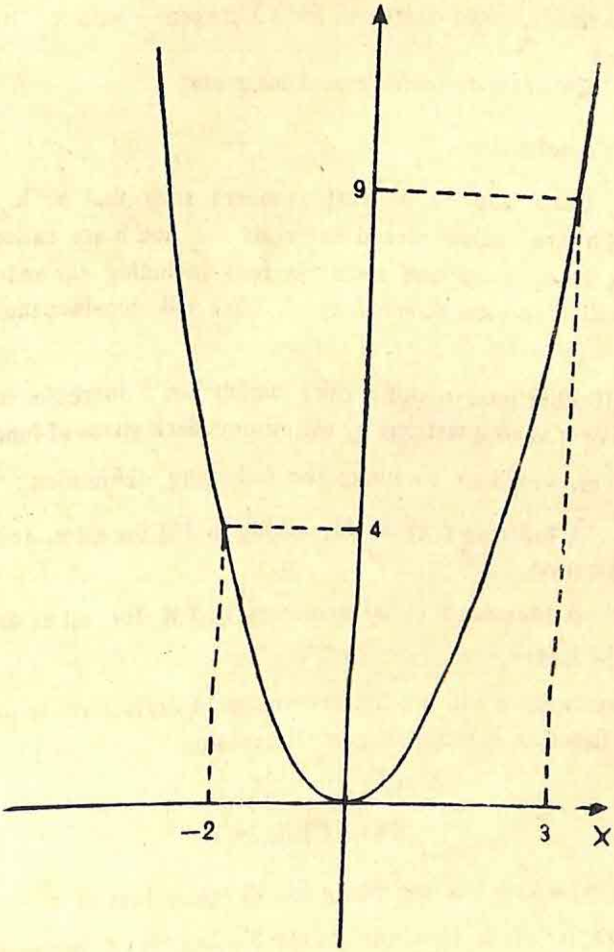


**Fig. 24.6:** Graph of $\phi(x)=x^2$ for all real $x$

[The reader is advised to show similarly that $\phi$ **is not decreasing.**]

**We thus see that $\phi(x)=x^2$ for all real $x$ is neither increasing nor decreasing.** (See Fig. 24.6)

However, if we consider only non-negative values of $x$ (where $\phi$ coincides with $f$), we find that $\phi$ is increasing. Similarly, $\phi$ is decreasing for $x\leqslant 0$.

*Sets* of the type $x\geqslant a$, $x\leqslant a$ are called* **closed intervals.** *$a$ is called an* **end point** *of these intervals.*

We recall from our study of elementary trigonometry that $\sin x$ increases for values of $x$ between 0 and $\frac{\pi}{2}$ and decreases for $x$ between $\frac{\pi}{2}$ and $\pi$. *Sets of the type $0\leqslant x\leqslant\frac{\pi}{2}$ and $\frac{\pi}{2}\leqslant x\leqslant\pi$ are also called* **closed intervals.**

We now make a definition.

**Definition 1 : Let $a$ and $b$ be real numbers such that $a<b$. Sets of the type $x\geqslant a$, $x\leqslant a$ and $a\leqslant x\leqslant b$ are called closed intervals. $a$ and $b$ are called their end-points. The corresponding sets $x>a$, $x<a$ and $a<x<b$ (not including the end-points) are called open intervals.** *We will denote an interval by* **J. We will consider the real line also to be an interval.**

Can you recall other intervals of $x$ over which $\sin x$ increases or decreases ? Let us see how we can answer such questions by examining **derivatives of functions.**

Before, however. we do so, we make the following definitions :

**Definition 2 : A function $f(x)$ is increasing in J if for all $x_1$ and $x_2$ in J such that $x_1<x_2$, we have $f(x_1)<f(x_2)$.**

**Definition 3 : A function $f(x)$ is decreasing in J if for all $x_1$ and $x_2$ in J such that $x_1<x_2$, we have $f(x_1)>f(x_2)$.**

In the next section,we will use the knowledge of derivatives to determine interval(*s*) over which a given function is increasing or decreasing.

## EXERCISE 24.1

1. Show that $g(x)=3x+1$ is increasing for all real values of $x$.
2. Draw a graph of a function $f(x)=3-2x$. Is it increasing or decreasing or neither ? Prove your assertion.

---

*If we were careful, we would use the set notation properly and write $\{x \mid x\geqslant a\}$ or $\{x \mid x\leqslant a\}$.

3. Prove that $f(x)=ax+b$, where $a$ and $b$ are constants and $a>0$, is an increasing function of $x$ for all real values of $x$.
4. Show that $\phi(x)=x^2$ for all real values of $x$ is neither increasing nor decreasing.

*5. Prove that $g(x)=\sin x$ is increasing in $0\leqslant x\leqslant \frac{\pi}{2}$.

### 24.3 Derivatives and Slopes

**We recall that the derivative at a point P of a curve is the slope of the tangent to the curve at the point P. Of course, the tangent to a curve at P is the limiting line of the secants PQ as Q moves towards P along the curve.** *We are, therefore, tempted to say that for an increasing function, the slope of the tangent (assuming that there is a tangent, of course !) must be positive.* But our intuition fails us here. Consider the graph of $f(x)=x^3$, for instance. (See Fig. 24.7) It is an increasing function.

However, at the origin,

$$f'(x)\Big]_{x=0}=3x^2\Big]_{x=0}=0$$

In other words, the slope of the tangent at the origin is not positive—it is zero.

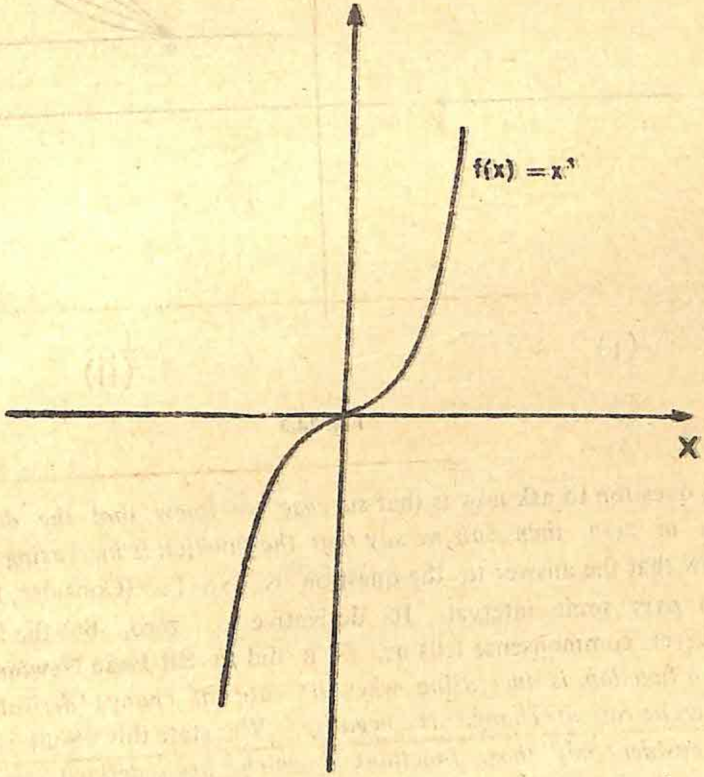
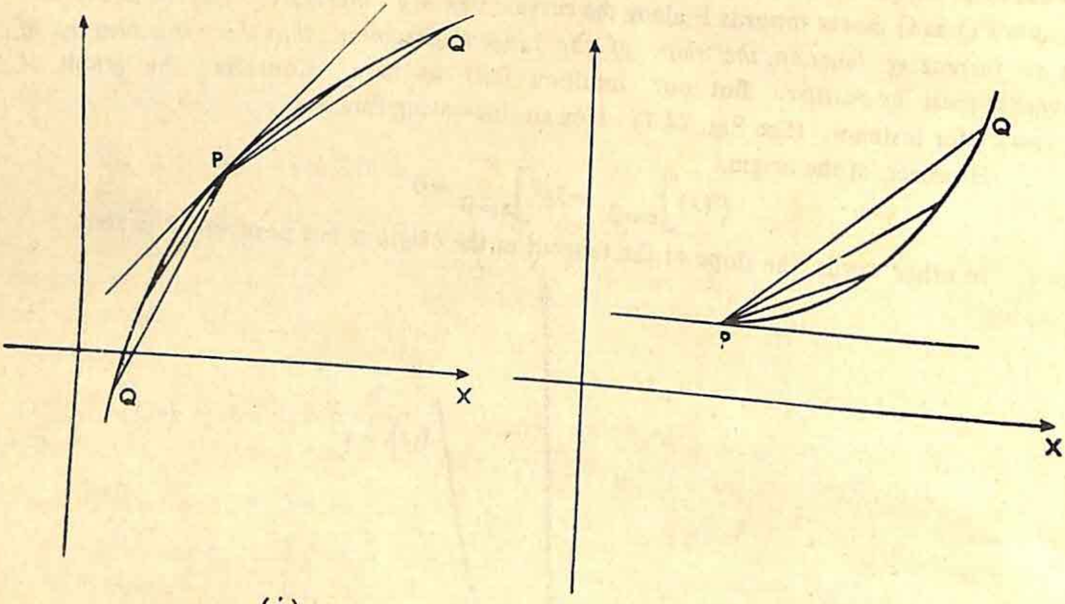


Fig. 24.7 : The tangent at (0, 0) to $f(x)=x^3$ coincides with the $x$-axis

However, we have the following situation :

If a function is increasing, then secants $PQ$ for all positions of the points $Q$ on the curve have positive slope. The limiting line of these secants has positive slope [See Fig. 24.8 (i)] or zero slope. [See Fig. 24.8 (ii)]. Thus for *increasing functions, slope of the tangent and hence, derivative is either positive or zero*. [It is left as an exercise for the reader to see that for *decreasing functions, slope of the tangent and hence. derivative is either negative or zero*.]



(i)

(ii)

Fig. 24.8

A natural question to ask now is that *suppose we know that the derivative of a function is positive or zero, then can we say that the function is increasing* ? A moment's reflection will show that the answer to the question is 'No !'. (Consider, for instance, a constant function over some interval. Its derivative is zero, but the function is not increasing.) However, commonsense tells us, as it did to **Sir Issac Newton (1642–1727 A.D.)**, to say that *a function is increasing when its rate of change (derivative) is positive and decreasing when its rate of change is negative*. We state this result as follows. *To begin with, we consider only those functions f which are defined everywhere and have derivative everywhere.*

**Theorem 12 :** **If $f'(x)>0$ for all x in J, then f(x) is increasing in J.**

**Theorem 13 :** **If $f'(x)<0$ for all x in J, then f(x) is decreasing in J.**

We will not attempt a formal proof of these theorems here. Their proofs make use of an important theorem called **The Mean Value Theorem (of Differential Calculus).** The famous French mathematician, **Joseph Louis Lagrange (1736—1813 A.D.),** who was responsible for a systematic development of mechanics using the methods of Calculus, provided a rigorous proof of The Mean Value Theorem. Some authors, therefore, refer to this theorem as **Lagrange's Mean Value Theorem.** It tells us that for the kind of functions we will mostly be dealing with, every **sacant PQ through any point P has at least one tangent parallel to it.** (See Figs. 24.9 and 24.10). The tangent will, of course, be at some (intermediate) point between *P* and *Q*.

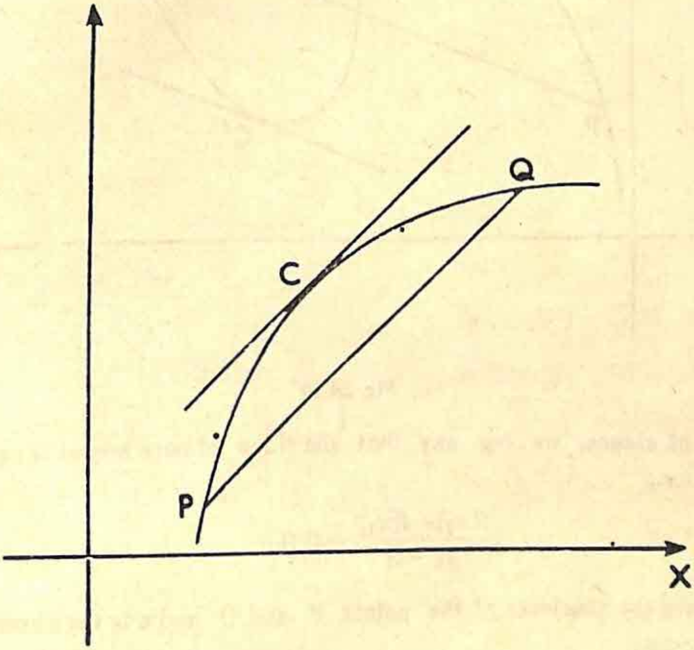


Fig. 24.9

This is intuitively quite clear. If we move the secant *PQ* to the left (or right) keeping it parallel to itself, there will be a situation (between positions where it cuts the curve in two points and where it does not meet the curve any more) where it will become a tangent to the curve. In Fig. 24.9, this occurs at the point $C$. What about in Fig. 24.10 ?

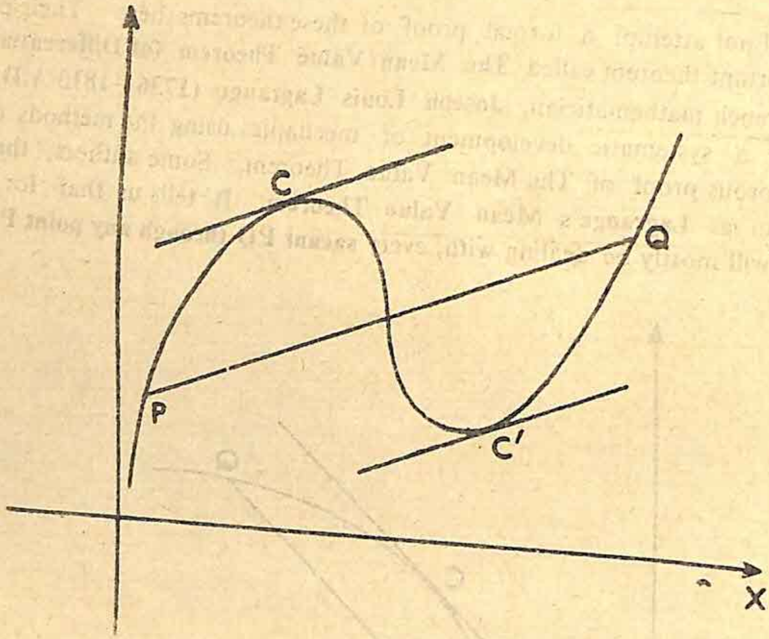


Fig. 24.10

**In terms of slopes, we can say that the slope of each secant is equal to the slope of some tangent, *i.e.*,**

$$\frac{f(x_2)-f(x_1)}{x_2-x_1}=f'(c)$$

**where $x_1$ and $x_2$ are the abscissae of the points P and Q and c is the abscissa of the point C such that $x_1<c<x_2$.**

**Remark :** We see that, in the conclusion of The Mean Value Theorem, we do not need the derivatives at the end-points. All we need is the derivative(s) at some intermediate point(s).

**It is now easy to prove Theorem 12.** If $f'(c)>0$, $f(x_2)-f(x_1)$ and $x_2-x_1$ must both simultaneously be positive or negative. This means that if $x_1<x_2$, then $f(x_1)<f(x_2)$ or if $x_1>x_2$, then $f(x_1)>f(x_2)$. **But this is precisely what we mean by an increasing function and this proves Theorem 12.**

**[The reader is advised to provide a similar heuristic proof to Theorem 13.]**

**We now have a test for determining intervals over which a given function is increasing or decreasing.** We state this test in the form of a theorem which we shall not bother to prove. As already remarked, *we consider only those functions $f(x)$ which have derivatives everywhere.*

**Theorem 14 : (i) If $f'(x)>0$ for all $x$ in some open interval, then $f(x)$ is increasing in the corresponding closed interval (including end-points).**

**(ii) If $f'(x)<0$ for all $x$ in some open interval, then $f(x)$ is decreasing in the corresponding closed interval (including end-points).**

We now consider some examples.

**Example 1 :** Find intervals in which

$$f(x)=x^2-2x$$

is increasing or decreasing.

**Solution :** We are given that

$$f(x)=x^2-2x$$

Thus, $$f'(x)=2x-2$$

Now $f'(x)>0$ when $2x-2>0$ *i.e.*, when $x>1$. Thus the function $f(x)=x^2-2x$ s increasing in the interval $x\geqslant 1$.

Similarly, we can show that $f(x)=x^2-2x$ is decreasing for $x\leqslant 1$.

**Example 2 :** Find intervals in which

$$f(x)=2x^3+x^2-20x$$

is increasing.

**Solution :** Since $f(x)=2x^3+x^2-20x$,

$$f'(x)=6x^2+2x-20$$

We need to determine for what values of $x$, $f'(x)>0$

Now, $f'(x)=2(3x^2+x-10)=2(3x-5)(x+2)$

Thus, $f'(x)>0$ if $(3x-5)$ and $(x+2)$ are both simultaneously positive or negative.

Now, when are $(3x-5)$ and $(x+2)$ both simultaneously positive ? They will both be simultaneously positive when $x>\frac{5}{3}$ and $x>-2$ *i.e.*, $x>\frac{5}{3}$.

And, when are $(3x-5)$ and $(x+2)$ both simultaneously negative ? When $x > \frac{5}{3}$ and $x < -2$ *i.e.*, $x < -2$.

Thus, $f(x)$ is increasing in $x \leqslant -2$ and in $x \geqslant \frac{5}{3}$

## EXERCISE 24.2

Show that the following functions are increasing in the indicated intervals :

1. $f(x) = x^2 - 8x,\ x \geqslant 4$
2. $f(x) = 6 + 9x - x^2,\ x \leqslant \frac{9}{2}$
3. $f(x) = x^3 - 27x,\ x < -3$ or $x > 3$

*4. $f(x) = x^7 + 8x^5 + 1$ for all values of $x$

[**Hint :** Prove that $f$ is increasing in $x \geqslant 0$ and also in $x \leqslant 0$]

5. $f(x) = \frac{1}{1+x^2},\ x \leqslant 0$

Determine for which values of $x$ the following functions are increasing and for which values they are decreasing :

6. $f(x) = x^4$

*7. $f(x) = x^8 + 6x^2$

8. $f(x) = 6x^2 - 2x + 1$
9. $f(x) = 2x^3 - 24x + 107$
10. $f(x) = x^4 - 4x$
11. $f(x) = \frac{1}{1+x^2}$

### 24.4 Derivatives and Graphs

We can now use the knowledge of derivatives to draw graphs. We consider some examples.

**Example 1 :** Draw a graph of

$$f(x)=x^2-2x$$

**Solution :** We have seen in Example 1, Section 24.3 that $f(x)$ is increasing in the interval $x \geqslant 1$ and decreasing in the interval $x \leqslant 1$.

Now, the pertinent questions are *how far does $f(x)$ increase and how far does $f(x)$ decrease*? In other words, does $f(x)$ go on increasing or does it stop somewhere? Similarly, does $f(x)$ go on decreasing or does it stop somewhere ? Let us see.

We write $f(x)$ as $f(x)=x(x-2)$

It is clear that for $x>3$, $x-2>1$ and since $x$ is positve, $x(x-2)>x$. Thus $f(x)$ becomes larger and larger as $x$ moves further away from the origin through positive values. (In fact, $f(x)$ always stays larger than $x$ for $x>3$). We say $\mathbf{f(x)\to\infty}$ **as** $\mathbf{x\to\infty}$ and read "*$f(x)$ approaches infinity as $x$ approaches infinity*". This means that, for $x>3$, the graph of $f(x)$ extends upward without any bound.

How about for negative $x$'s ? We see that if $x<0$, $2-x>1$ and since $x$ is negative, $-x(2-x)>-x$. Thus $f(x)$ becomes larger and larger as $x$ moves further away from the origin through negative values. We say $\mathbf{f(x)\to\infty}$ **as** $\mathbf{x\to-\infty}$ and read "*$f(x)$ approaches infinity as $x$ approaches negative infinity*." This means that, for $x<0$, the graph of $f(x)$ extends upward without any bound. Note that $f(x)$ is decreasing in $x \leqslant 1$.

Finally since $f'(x)=0$ at $x=1$, we have a **horizontal tangent at** $\mathbf{x=1}$.

We construct a table of values of $x$ and $y$.

| $x$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ | $4$ |
|---|---|---|---|---|---|---|---|
| $y$ | $8$ | $3$ | $0$ | $-1$ | $0$ | $3$ | $8$ |

We join the points for which $x \geqslant 1$ by a steadily rising curve and the points for which $x \leqslant 1$ by a steadily falling curve. Recall that at the point where $x=1$, we will have

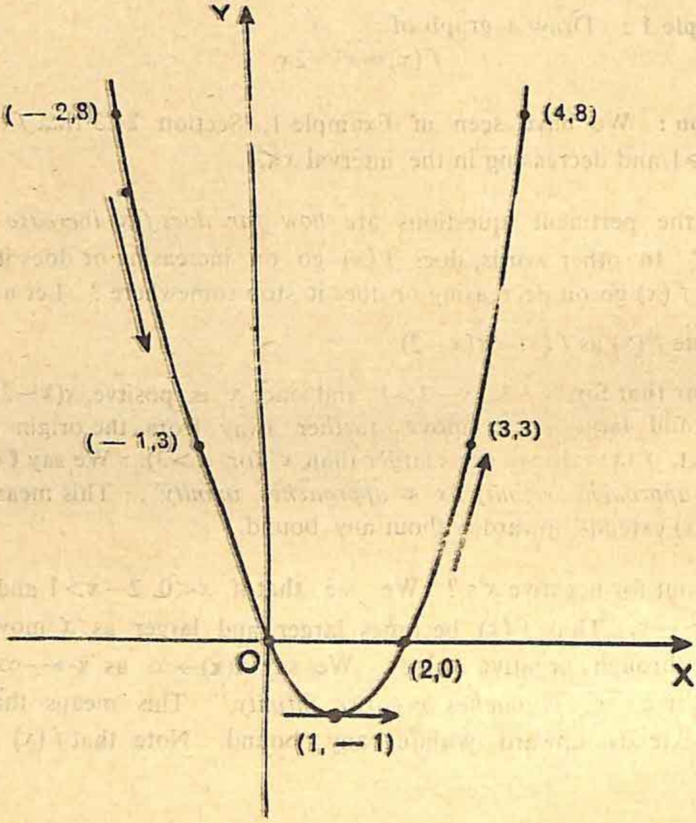


Fig. 24. 11: Graph of $f(x)=x^2-2x$

a horizontal tangent. We have a graph of $f(x)=x^2-2x$ as shown in Fig. 24.11.

**Example 2 :** Draw a graph of

$$f(x)=x^3-3x$$

**Solution :** Since $f(x)=x^3-3x$,

we have $f'(x)=3x^2-3=3(x^2-1)$

For what values of $x$ is $f'(x)>0$ ? It is obvious that $f'(x)>0$ if $x^2>1$ *i.e.*, if $x>1$ or $x<-1$.

And that $f'(x)<0$ if $x^2<1$ *i.e.*, if $|x|<1$.

Thus *for values of x such that $-1\leqslant x\leqslant 1$, the graph of f(x) is* **decreasing**. *And for $x\geqslant 1$ as well as for $x\leqslant 1$, the graph of f(x) is* **increasing**.

**But how far does f(x) increase ?** Let us see.
We rewrite $f(x)$ as $f(x)=x(x^2-3)$

It is clear that for $x>2$, $x^2-3>1$ and since $x$ is positive, $x(x^2-3)>x$. Thus **$f(x)\to\infty$ as $x\to\infty$.** This means, that **the graph of f(x). for $x>2$, extends upward without any bound.**

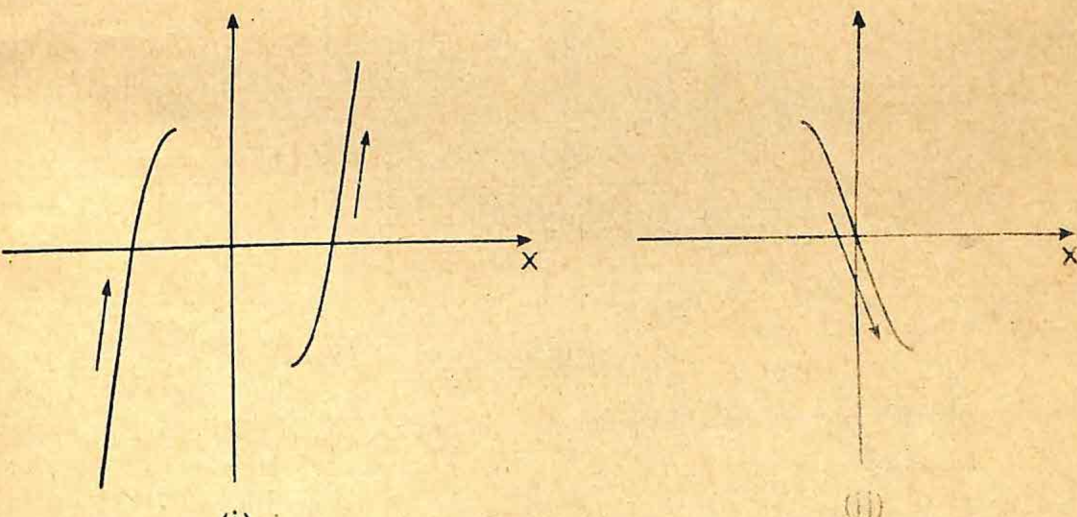
Also for $x<-2$, $x^2-3>1$ and since $x$ is negative, $x(x^2-3)<x$. Thus $f(x)$ becomes smaller and smaller (*i.e.*, negative and farther and farther away from zero) as $x$ moves farther away from the origin through negative values. We write **$f(x)\to-\infty$ as $x\to-\infty$** and read "$f(x)$ *approaches negative infinity as x approaches negative infinity.*"

**The graph of f(x), therefore, for $x<-2$, extends downward without any bound.**

Finally, since $f'(x)=0$ at the points $x=1$ and $x=-1$, **we have horizontal tangents at these points.**

We construct a table of values of $x$ and $y$ and plot a graph of $f$ in two stages.

| $x$ | $-3$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ |
|---|---|---|---|---|---|---|---|
| $y$ | $-18$ | $-2$ | $2$ | $0$ | $-2$ | $2$ | $18$ |



(i)

(ii)

Graph of $f(x)=x^3-3x$ for $x\geqslant 1$ and for $x\leqslant -1$

Graph of $f(x)=x^3-3x$ for $-1\leqslant x\leqslant 1$

Fig. 24.12

The values of $x$ for which the curve is rising and the resulting values of $y$ are plotted in Fig. 24.12 (*i*). The values of $x$ for which the curve is falling and the resulting values of $y$ are plotted in Fig. 24.12 (*ii*).

Finally, we show the two stages together and a graph of $f(x)=x^3-3x$ is shown in Fig. 24.13.

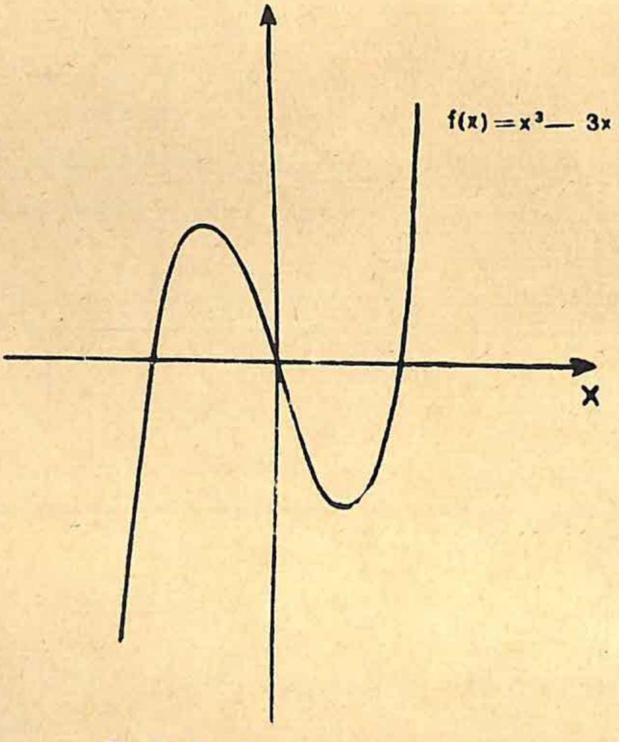


**Fig. 24.13 :** Graph of $f(x)=x^3-3x$

## EXERCISE 24.3

Sketch a graph of each of the following functions :

1. $y=x^3$
2. $y=x^2-x$
3. $y=x^2+x+1$
4. $y=x^2-1$

5. $y=(x-1)^2$
6. $y=2x^3+12x-9x^2$
7. $y=x^3+x$

*8. $y=x(x-1)(x-2)$

9. $y=x^4-2x^2$

*10. $y=\frac{1}{1+x^2}$

## 24.5 Continuity

In drawing graphs in Section 24.4, we never used '*broken*' curves of the type shown, for instance, in Fig. 24.14. We will explain the reason in this section.

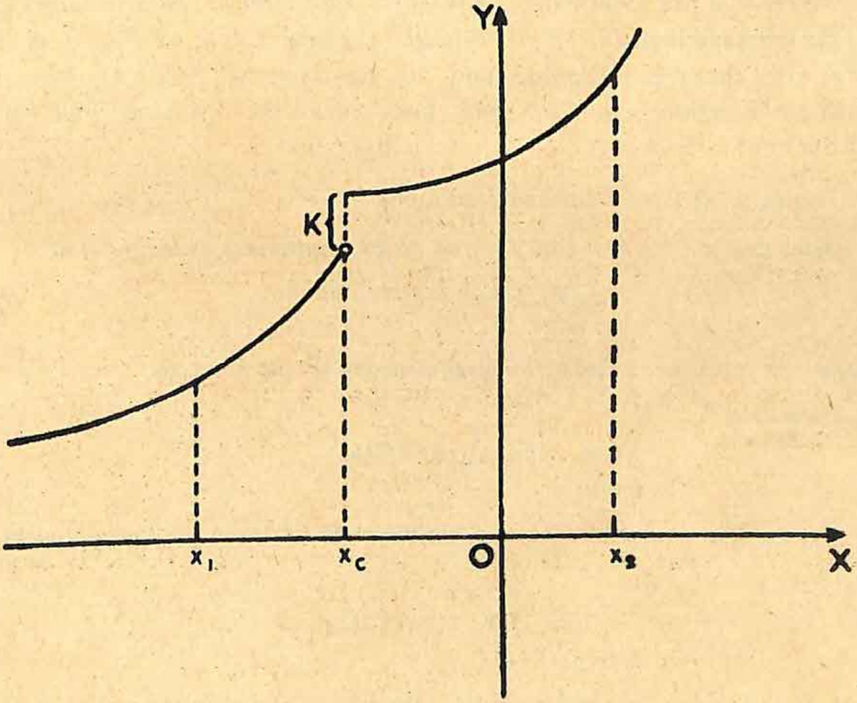


**Fig. 24.14 : Graph of y=f(x)**

The graph of $y=f(x)$ in Fig. 24.14 has a **'jump'** of a certain height, say K, at the point $x_0$ in the sense that *if we were moving along the graph from left to right, there is a break at* $x_0$ *and we would have to lift the pencil in moving from one side of* $x_0$ *to its other side.*

In other words, if we approach $x_0$ from the R.H.S. (*i,e.*, through positive values of $\triangle x$), then $f(x_0+\triangle x)$ approaches $f(x_0)$. We write this as

$$\lim_{\triangle x\to 0+} f(x_0+\triangle x)=f(x_0)\cdot$$

and call it the **right-hand limit of f(x) at $x_0$.**

However, if we approach $x_0$ from the L.H.S. (*i.e.*, through negative values of $\triangle x$), then $f(x_0+\triangle x)$ approaches $f(x_0)$—K. We write this as

$$\lim_{\triangle x\to 0-} f(x_0+\triangle x)=f(x_0)-K$$

and call it the **left-hand limit of f(x) at $x_0$.**

We see that *the* **right-hand limit** (*i.e., limit through positive values of* $\triangle x$) *is not equal to the* **left-hand limit** (*i.e., limit through negative values of* $\triangle x$) *at the point* $x_0$. It should be noted that this is the case **only** at the point $x_0$. For every other point in the domain of the function, the right-hand limit equals the left-hand limit and equals the value of the limit.

This motivates the following definitions :

**Definition 4 :** A function $f$ is said to be **continuous at the point $x_0$** if

$$\lim_{\triangle x\to 0} f(x_0+\triangle x)=f(x_0)$$

Otherwise, the function is said to be **discontinuous at the point $x_0$**.

**Definition 5 :** If

$$\lim_{\triangle x\to 0+} f(x_0+\triangle x)=f(x_0),$$

we say that the function is **continuous from the right** (or **right-continuous**) **at the point $x_0$**.

If

$$\lim_{\triangle x\to 0-} f(x_0+\triangle x)=f(x_0),$$

we say that the function is **continuous from the left** (or **left-continuous**) **at the point $x_0$.**

It is clear that for a function to be continuous at a point $x_0$, it must be right-continuous as well as left-continuous at the point $x_0$. *Also, if a function is defined in a closed interval, then we can only talk about one sided continuity at an end-point and that will be taken to mean the* **continuity at an end-point**.

**It is instructive to note that Lagrange's Mean Value Theorem as well as Theorem 14 are valid for functions f which are defined and continuous on a closed interval J and have a derivative at each point in the corresponding open interval.**

The graph of $y=f(x)$ in the Fig. 24.14 is right-continuous but not left-continuous. Can you give the reasons, why ?

It is clear that the functions that we graphed in Section 24.4 are continuous (*i.e.*, **continuous at every point of the domain**), and this is why their graphs have no *breaks* or *jumps*.

We give some examples of discontinuous functions and draw their graphs.

**Example 1 :** Draw a graph of

$$f(x)=\begin{cases} 2 \text{ for } x<0 \\ x \text{ for } x\geqslant 0 \end{cases}$$

Prove that the function is discontinuous at the origin.

**Solution :** The graph of $f(x)$ is rather easy to draw. It is shown in Fig. 24.15. Obviously, the graph has a jump at the point $x_0=0$. Let us examine the right-hand and the left-hand limit at $x=0$. We have, for $\triangle x>0$, $f(0+\triangle x)=\triangle x$.

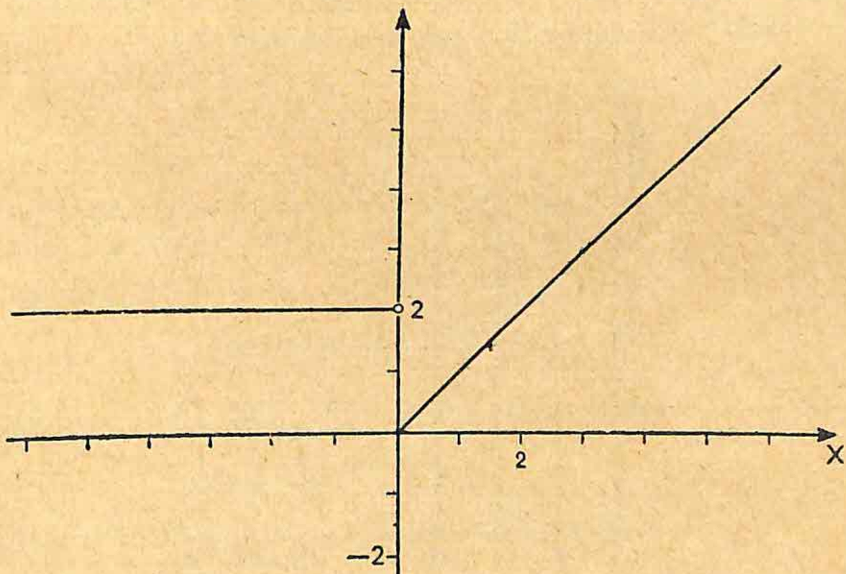


**Fig. 24.15 :** Graph of $f(x)=2$, $x<0$; $f(x)=x$, $x\geqslant 0$

Thus, $$\lim_{\Delta x\to 0+} f(0+\Delta x)=0$$

And for $\Delta x<0$, $f(0+\Delta x)=2$

Thus, $$\lim_{\Delta x\to 0-} f(0+\Delta x)=2$$

We see, therefore, that

$$\lim_{\Delta x\to 0+} f(0+\Delta x)\neq \lim_{\Delta x\to 0-} f(0+\Delta x)$$

Thus $f(x)$ is discontinuous at $x=0$.

[The reader is advised to show that $f(x)$ is right continuous at $x=0$.]

**Example 2 :** The *postage-function* for letters, within India, is given below (as on 2/09/78). If $P(x)$ is the postage, in paise, of a letter whose weight, in grams, is $x$, then

$$P(x)=\begin{cases} 25, & 0<x\leqslant 10 \\ 40, & 10<x\leqslant 20 \\ 55, & 20<x\leqslant 30 \end{cases}$$

Draw its graph.

**Solution :** Since $x$ cannot take zero or negative values, it is clear that $P(x)$ is not defined for $x\leqslant 0$. The graph of $P(x)$ is given in Fig.24.16.

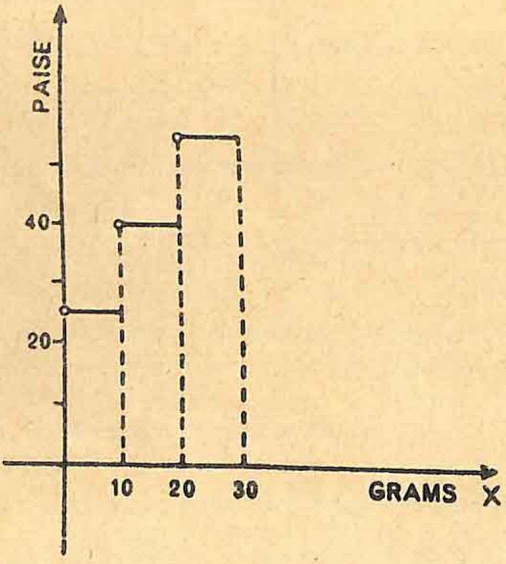


Fig. 24.16. Graph of the postage-function

[The reader is advised to show that P($x$) is left continuous at the points $x=10$, 20 and 30.]

It is not our intention to go deeply into continuous and discontinuous functions and types of discontinuities. We will rely heavily on our intuition and observation to decide when a function is continuous and when not.

## 24.6 Continuity and Differentiability

We now examine the relationship between the existence of the derivative at a point and continuity at that point. We have the following important theorem which tells us that

**Theorem 15 : If a function has a derivative at a point, it is continuous at that point.**

The proof of this theorem is rather simple. Since the function has a derivative at a point, say $x_0$,

$$\lim_{\Delta x \to 0} \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}=f'(x_0) \tag{1}$$

We wish to show that the function is then continuous at $x_0$; *i.e.*,

$$\lim_{\Delta x \to 0} f(x_0+\Delta x)=f(x_0)$$

or,

$$\lim_{\Delta x \to 0} [f(x_0+\Delta x)-f(x_0)]=0 \tag{2}$$

How shall we proceed ? We recall that we have done a similar exercise in the proof of Theorem 6 in Section 21.2.

We rewrite L.H.S. of (2) as

$$\lim_{\Delta x \to 0} \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}.\Delta x$$

$$=\lim_{\Delta x \to 0} \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}\cdot \lim_{\Delta x \to 0} \Delta x \qquad \text{(Why?)}$$

$$=f'(x_0)\cdot 0=0$$

This proves the result.

Thus every **differentiable* function is continuous**. In particular, *polynomial func- tions are continuous at every point.*

---

*A function is **differentiable** if it has a derivative at every point of its domain.

Let us now examine the **converse** of this theorem. Is every continuous function differentiable ? The answer to this question is 'No' ! Let us consider the **absolute value function $\mathbf{f(x) = |x|}$**. Its graph is shown in Fig. 24.17.

It is easy to see that $f(x)$ is continuous at the origin. Is it differentiable at $x_0=0$ ? Let us see.

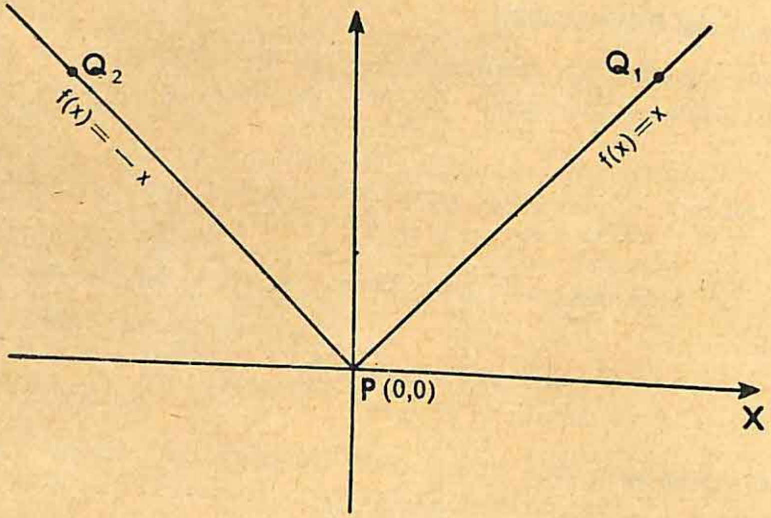


**Fig. 24.17 :** Graph of $f(x) = |x|$

We need to evaluate, if possible,

$$\lim_{\Delta x \to 0} \left[ \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x} \right]$$

where $x_0=0$ and

$$f(x) = \begin{cases} x, & x \geqslant 0 \\ -x, & x \leqslant 0 \end{cases}$$

Let $\Delta x \to 0$ through positive values. Then $f(x_0+\Delta x) = \Delta x$

Hence, $\dfrac{f(x_0+\Delta x)-f(x_0)}{\Delta x} = \dfrac{\Delta x - 0}{\Delta x} = 1$

Thus, $\displaystyle \lim_{\Delta x \to 0+} \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x} = 1$

Now let $\Delta x \to 0$ through negative values. Then

$$f(x_0+\Delta x) = -\Delta x$$

$$\text{Hence, } \frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}=\frac{-\triangle x-0}{\triangle x}=-1$$

$$\text{Thus, } \lim_{\triangle x\to 0-}\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}=-1$$

These limits are called, respectively, the **right-hand derivative** and **left-hand derivative of the function f(x) at $x_0$.** We have

**Right-hand Derivative at the point $x_0=f'(x_0+)$**

$$=\lim_{\triangle x\to 0+}\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}$$

**Left-hand Derivative at the point $x_0=f'(x_0-)$**

$$=\lim_{\triangle x\to 0-}\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}$$

Since $f'(x_0+)\neq f'(x_0-)$, at $x_0=0$, **the function $|x|$ is not differentiable at the origin.**

[That the function is not differentiable at the origin is also obvious geometrically. We recall from Section 20.2 that secants PQ must approach the tangent at P for positions of Q on both sides of P on the curve (assuming, of course that P is not an end-point). In our example, the secants $PQ_1$ for positions of $Q_1$ on $f(x)=x$, $x\geqslant 0$ approach the line $y=x$; while the secants $PQ_2$ for positions of $Q_2$ on $f(x)=-x$, $x\leqslant 0$ approach the line $y=-x$. The secants, therefore, do not approach a common line as Q approaches P along the curve.]

**Differentiability of a function at a point in an open interval, therefore, requires that both the right-hand and left-hand derivatives are equal at that point. Geometrically, it ensures that the graph of the function is *smooth* at that point. The inequality of the right-hand and left-hand derivatives at a point geometrically means that the curve has a *corner* at that point. If the function is defined on a closed interval, then we can only talk about one-sided derivatives at an end-point and that will be taken to mean the derivative at an end point.**

We consider an example.

**Example 1 :** Show that the function $f$ given by

$$f(x)=\begin{cases} x-1, & x<2 \\ 2x-3, & x\geqslant 2 \end{cases}$$

is continuous at every point. Draw a graph of the function.

**Solution** : For $x<2$,

$$f(x)=x-1$$

Thus, $$f'(x)=1$$

Since the function is differentiable for $x<2$, it is continuous for $x<2$. Let us leave the point $x=2$ out of our discussion for a moment. For $x>2$, the function is also differentiable and hence continuous. So we have continuity for every point of the domain, except for $x=2$, for which we are not sure yet.

We must examine the right-continuity and left-continuity of the function at $x=2$. Let us do so.

Let $\Delta x$ be positive.

Then, $f(2+\Delta x)=2(2+\Delta x)-3=2\Delta x+1$

Thus, $\lim\limits_{\Delta x\to 0+} f(2+\Delta x)=1=f(2)$

Now let $\Delta x$ be negative.

We have, $f(2+\Delta x)=2+\Delta x-1=1+\Delta x$

Thus, $\lim\limits_{\Delta x\to 0-} f(2+\Delta x)=1=f(2)$

We see, therefore, that

$$\lim_{\Delta x\to 0+} f(2+\Delta x)=f(2)=\lim_{\Delta x\to 0-} f(2+\Delta x)$$

Thus $f$ is continuous at $x=2$. This proves the result.

[The reader is advised to prove that $f$ is not differentiable at $x=2$.]

Graph of f (x) is shown in Fig. 24.18.

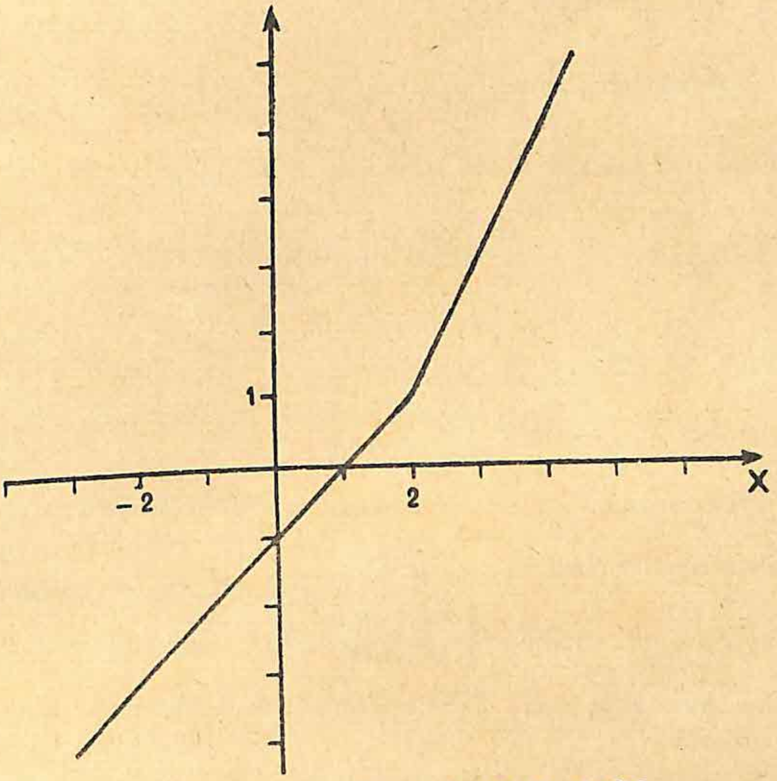


Fig. 24.18 : Graph of $f(x)=x-1,\ x<2\ ;\ 2x-3,\ x\geqslant 2$

## EXERCISE 24.4

For each of the following functions, find the point(s) of discontinuity, if any, and draw a graph

1. $f(x)=\begin{cases} -1, & x\leqslant 0 \\ +1, & x>0 \end{cases}$

2. $f(x)=|x-1|$

*3. $f(x) = |x| + |x-1|$

Show that each of the following functions is continuous at the point(s) indicated.

4. $f(x)=x^2-7x+3$ ; $x=1, 3$

5. $f(x)=x^{104}-\dfrac{x}{3}$ ; $x=5$

6. Prove that $x+\sqrt{x}$ is not continuous at the origin.

7. Prove that $f(x)=|x|$ is continuous at the origin.

8. Prove that the function

$$f(x)=\begin{cases} 3x-2, & x\leqslant 0 \\ x+1, & x>0 \end{cases}$$

is not continuous at $x=0$. Sketch a graph of the function.

9. Show that the function

$$f(x)=\begin{cases} -x^2, & x\leqslant 0 \\ x^2, & x>0 \end{cases}$$

is continuous at $x=0$. Sketch a graph of the function. Is it differentiable at $x=0$ ?

10. Find the right-hand derivative and the left-hand derivative of

$$f(x)=\begin{cases} x-1, & x<2 \\ 2x-3, & x\geqslant 2 \end{cases}$$

at the point $x=2$. Hence show that $f$ is not differentiable at $x=2$.

11. Is the function

$$f(x)=\begin{cases} \dfrac{1}{x}, & x\neq 0 \\ 3, & x=0 \end{cases}$$

continuous at $x=0$ ? Give reasons for your answer.

*12. If $f$ and $g$ are both continuous at $x=x_0$, show that the function $h$ defined by $h(x)=f(x)+g(x)$ is also continuous at $x=x_0$. What about the function $\phi$ defined by $\phi(x)=f(x)\,g(x)$ ?

*13. $f(x)=\dfrac{3x^3-2x^2-1}{x-1}$, $x\neq 1$. What value should be assigned to $f(1)$ so that $f(x)$ is continuous at $x=1$ ?

## 24.7 Key Concepts

Increasing function
Decreasing function
Intervals—end-points
Lagrange's Mean Value Theorem
Continuity
Right continuous and left continuous
Right-hand and left-hand limits
Discontinuity at a point
Differentiability
Right-hand and left-hand derivatives

## 24.8 Suggestions For Further Reading

*An excellent discussion of using Calculus to draw graphs is found in* [2], *Sectior 18.7.*

*The reader is also referred to* [1], *Section 18.7 and* [1], *Section 21.9.*

*A good intuitive discussion, supplemented by rigour in appendices is found in*

[1] Lipman Bers : **Calculus, Volume I.**
IBH Publishing Company, Bombay (India). 1973.

# UNIT XXV

# DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

*We learn how to find derivatives of trigonometric functions. We also study how to graph the functions sin x and cos x.*

## 25.1 Introduction

So far we have learnt how to differentiate polynomials, their quotients and a few other types of functions. We now learn how to find derivatives of trigonometric functions. However, before we do so, we need a couple of basic results.

## 25.2 Two Basic Results

We will require the following two results in finding the derivatives of trigonometric functions :

(*i*) $\lim_{t \to 0} (\sin t) = 0$

(*ii*) $\lim_{t \to 0} \left[\frac{\sin t}{t}\right] = 1$

Both these results make use of a result which we have often used before without formally saying so. We now state this result formally.

**Theorem 16:** **If f, g and h are functions such that**

$$f(x) \leqslant g(x) \leqslant h(x)$$

**for all x (except possibly at $x_0$) and if**

$$\lim_{\Delta x \to 0} f(x_0+\Delta x)=\lim_{\Delta x \to 0} h(x_0+\Delta x)=l \text{ (say)},$$

**then,**

$$\lim_{\Delta x \to 0} g(x_0+\Delta x)=l$$

Roughly speaking, the theorem tells us that *if a function can be* "**sandwiched**" *between two other functions, each of which approaches the same limit l, say, as x approaches $x_0$, then the sandwiched function also approaches the same limit l as x approaches $x_0$.* For obvious reasons, we call* it the "**Sandwich Theorem**".

We now prove result (*i*) below :

**Theorem 17 : If t is measured in radians, then**

$$\lim_{t \to 0} (\sin t)=0$$

**Proof :** Let us consider a unit circle C with its centre at the origin. Let A be the point (1, 0) on the circle and let P be a point on the circle such that ∠AOP, measured in radians, is between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. (See Fig. 25.1)

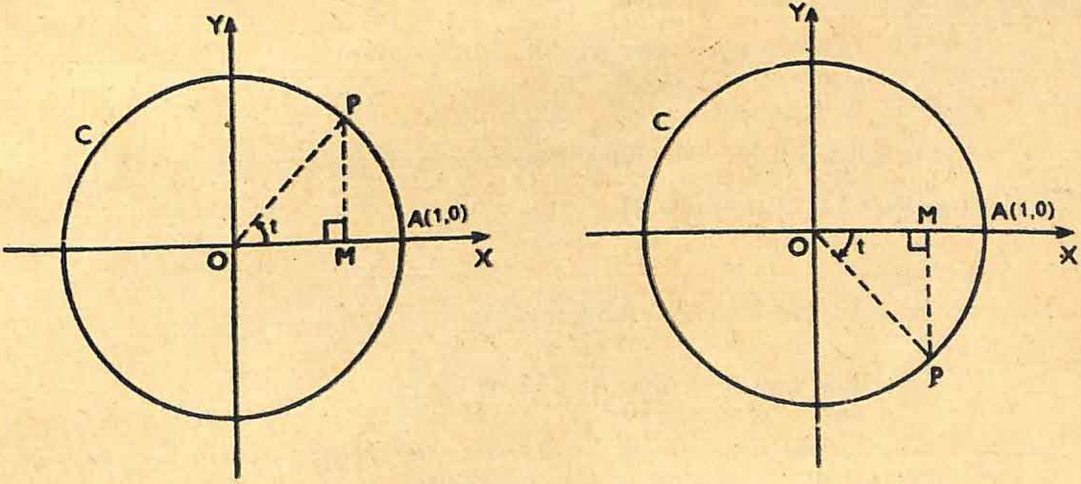


Fig. 25.1

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*Some authors call it the "**Pinching Theorem**". Similar theorems can also be stated for right-hand and left-hand limits.

If M is the foot of the perpendicular from P to the $x$-axis, then we recall that

$$| \text{MP} | = | \sin t |$$

We also recall that the length of the arc AP is $| t |$. It is clear from Fig. 25.1 that

$$| \mathbf{MP} | \leqslant | \widehat{\mathbf{AP}} |$$

Thus, $0 \leqslant | \sin t | \leqslant | t |$

Hence, $- | t | \leqslant \sin t \leqslant | t |$

Now as $t \to 0$, so does $| t |$. Thus, by the **Sandwich Theorem**,

$$\lim_{t \to 0} \sin t = 0$$

This proves the desired result.

It follows rather easily from this theorem that

**Corollary* 1 : If 'a' is a real number such that $a \neq 0$, then**

$$\lim_{t \to 0} (\sin at) = 0$$

[**Hint :** If $a \neq 0$ and if $t \to 0$, then $at \to 0$ and conversely, *if* $at \to 0$, then $t \to 0$. By letting, $at = u$ in Theorem 17, we have

$$\lim_{t \to 0} (\sin at) = \lim_{at \to 0} (\sin at) = \lim_{u \to 0} (\sin u) = 0]$$

We now have another important result as a consequence of Theorem 17.

**Corollary 2 : $\lim_{t \to 0} (\cos t) = 1$**

Since $\cos t = 1 - 2 \sin^2 \frac{t}{2}$, it follows that

$$\lim_{t \to 0} (\cos t) = \lim_{t \to 0} \left[ 1 - 2 \sin^2 \frac{t}{2} \right]$$

---

*Since $s^\circ = \frac{\pi s}{180}$ radians, Corollary 1 also indicates that

$$\lim_{s \to 0} \sin s^\circ = 0$$

$$= 1 - 2 \lim_{t \to 0} \left[ \sin^2 \frac{t}{2} \right]$$

$$= 1 - 2 \left[ \lim_{t \to 0} \left( \sin \frac{t}{2} \right) \right]^2$$

$$= 1 - 2(0)^2 = 1$$

This proves the result

Finally, we have a result based on Corollary 2.

**Corollary 3 :** If '$a$' is a real number such that $a \neq 0$, then

$$\lim_{t \to 0} (\cos at) = 1$$

**Remark :** We note that we have proved

$$\lim_{t \to 0} (\sin t) = 0 = \sin (0)$$

and

$$\lim_{t \to 0} (\cos t) = 1 = \cos (0)$$

This, of course, means that the 'sine' and 'cosine' are continuous at zero. We will see later that $\sin t$ and $\cos t$ are continuous for other values of $t$ as well. (See also Question 6, Exercise 24.4)

Next, we prove result (*ii*).

**Theorem 18 : If t is measured in radians, then**

$$\lim_{t \to 0} \frac{\sin t}{t} = 1$$

As $t$ approaches zero, it is clear that $\sin t$ approaches zero. That $\frac{\sin t}{t}$ approa ches 1 as $t$ approaches zero is suggested by the following table : (An electronic calculator has been used.)

| $t$ | $\sin t$ | $\frac{\sin t}{t}$ |
|---|---|---|
| 2 | 0.909297427 | 0.4546487135 |
| 1 | 0.841470984 | 0.841470984 |
| 0.5 | 0.479425538 | 0.958851076 |
| 0.4 | 0.389418341 | 0.9735458525 |
| 0.3 | 0.295520205 | 0.98506735 |
| 0.2 | 0.19866933 | 0.99334665 |
| 0.1 | 0.0998334156 | 0.998334156 |
| 0.09 | 0.089878549 | 0.9986505444 |
| 0.05 | 0.0499791692 | 0.999583384 |
| 0.04 | 0.0399893329 | 0.9997333225 |
| 0.03 | 0.0299954992 | 0.9998499733 |
| 0.02 | 0.0199986665 | 0.999933325 |
| 0.01 | 0.0099998324 | 0.99998324 |
| 0.005 | 0.0049999789 | 0.99999578 |
| 0.001 | 0.0009999988 | 0.9999988 |

What about $\frac{\sin t}{t}$ for negative values of $t$? We recall that if $t>0$, $\sin(-t) = -\sin t$.

Thus $\frac{\sin(-t)}{-t} = \frac{\sin t}{t}$, and we do not really need to construct another table for negative values of $t$.

Now, from the table, we guess that $\frac{\sin t}{t}$ **approaches 1 as t approaches zero.** We prove that it is indeed so. For the proof that we give, we will need the formula for the area of a sector of a circle. We recall that *area of a sector, which makes an angle of* $t$

*radians at the centre is*

$$\frac{\pi r^2 t}{2\pi} = \frac{1}{2}\ r^2 t$$

*where r is the radius of the circle.*

Let us now consider a unit circle C with its centre at the origin. Let A be the point (1, 0) on the circle and let P be a point on the circle such that $\angle$AOP is $t$ radians

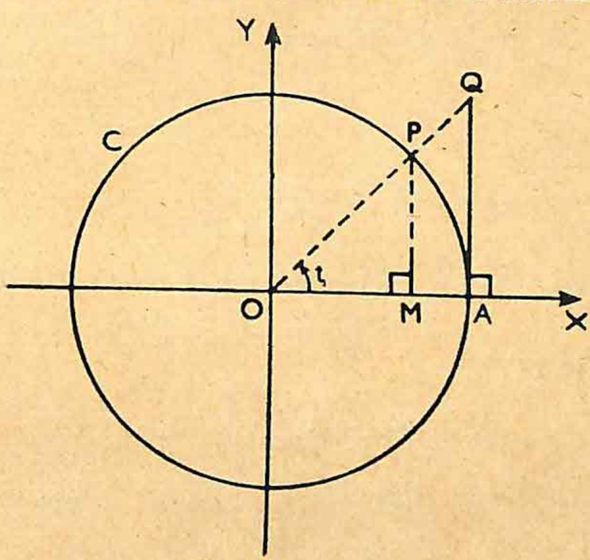

Fig. 25.2

$\left(0<t<\frac{\pi}{2}\right)$. $\Big[$Since, we have to let $t$ approach zero, it is reasonable to assume that $t$ is between 0 and $\frac{\pi}{2}$. $\Big]$

We draw PM$\perp$OA. At A, we draw a tangent to the circle to meet OP produced at Q. Then QA$\perp$OA. (Why ?) (See Fig. 25.2).

It is clear from Fig. 25.2 that the area of the sector OAP lies between the area of the right triangles OMP and OAQ. We have

$$\frac{1}{2}\text{OM}\times\text{MP} < \frac{(1)^2\ t}{2} < \frac{1}{2}\ \text{OA}\times\text{AQ} \tag{1}$$

But OA$=$OP$=1$, OM$=\cos t$, PM$=\sin t$ and AQ$=\tan t$.
Thus (1) becomes

$$\frac{1}{2}\cos t\times\sin t < \frac{1}{2}t < \frac{1}{2}\tan t$$

*i.e.,* $\cos t < \dfrac{t}{\sin t} < \dfrac{1}{\cos t}$ (Why ?) **(2)**

But we need $\lim\limits_{t \to 0} \left( \dfrac{\sin t}{t} \right)$. We, therefore, rewrite (2) as,

for $0 < t < \dfrac{\pi}{2}$

$$\frac{1}{\cos t} > \frac{\sin t}{t} > \cos t$$

*i.e., for* $0 < t < \dfrac{\pi}{2}$,

$$\cos t < \frac{\sin t}{t} < \frac{1}{\cos t} \tag{3}$$

We recall that $\dfrac{\sin(-t)}{-t} = \dfrac{\sin t}{t}$ and $\cos(-t) = \cos t$. **Thus, it is clear that** *for* $-\dfrac{\pi}{2} < t < 0$ *also, we have*

$$\cos t < \frac{\sin t}{t} < \frac{1}{\cos t} \tag{4}$$

Now $\lim\limits_{t \to 0} (\cos t) = 1$. Thus, $\lim\limits_{t \to 0} \left( \dfrac{1}{\cos t} \right) = 1$ (Why ?)

By **Sandwich Theorem**, therefore, we have that **if t is measured in radians, then**

$$\lim_{t \to 0} \left( \frac{\sin t}{t} \right) = 1$$

We consider some examples.

**Example 1** : **Show that** $\lim\limits_{t \to 0} \dfrac{\sin at}{t} = a$

**Solution :** Let $a = 0$ Setting $at = u$, we have if $t \to 0$, $at \to 0$, and if $at \to 0$, $t \to 0$.

Thus, $\lim\limits_{t \to 0} \dfrac{\sin at}{t} = \lim\limits_{at \to 0} \dfrac{\sin at}{t} = \lim\limits_{u \to 0} \dfrac{\sin u}{\frac{u}{a}}$

$$= a \lim_{u \to 0} \frac{\sin u}{u}$$

$$= a$$

Now let $a=0$. Then $\sin at$ is zero for all non-zero $t$, and hence $\frac{\sin at}{t}$ is zero for all non-zero $t$. Thus

$$\lim_{t\to 0} \frac{\sin at}{t}=0$$

We have thus shown that

$$\lim_{t\to 0} \frac{\sin at}{t}=a$$

Let us now find

$$\lim_{s\to 0} \frac{\sin s^\circ}{s}$$

We recall that $s^\circ = \frac{\pi s}{180}$ radians

Thus, $$\lim_{s\to 0} \frac{\sin s^\circ}{s} = \lim_{s\to 0} \frac{\sin \frac{\pi s}{180}}{s} = \frac{\pi}{180}$$

**Example 2 :** Show that $\lim_{t\to 0} \frac{\tan t}{t}=1$

**Solution :** We have $\lim_{t\to 0} \frac{\tan t}{t} = \lim_{t\to 0} \frac{\sin t}{t \cos t}$

$$=\left[\lim_{t\to 0} \frac{\sin t}{t}\right]\left[\lim_{t\to 0} \frac{1}{\cos t}\right]$$
$$=1$$

**Example 3 :** Prove that $\lim_{x\to 0} \frac{1-\cos x}{x}=0$

**Solution :** We recall that $1-\cos x=2\sin^2 \frac{x}{2}$

Thus, $$\lim_{x\to 0} \frac{1-\cos x}{x} = \lim_{x\to 0} \left(\frac{2\sin^2 \frac{x}{2}}{x}\right)$$

$$= \lim_{x\to 0}\left[2\left\{\frac{\sin\left(\frac{x}{2}\right)}{\frac{x}{2}}\right\}^2 \frac{x}{4}\right]$$

$$= \frac{1}{2}\left[\lim_{x\to 0}\frac{\sin\left(\frac{x}{2}\right)}{\frac{x}{2}}\right]^2\left[\lim_{x\to 0} x\right]$$

But $\lim\limits_{x\to 0}\dfrac{\sin\left(\frac{x}{2}\right)}{\frac{x}{2}}=1$ and $\lim\limits_{x\to 0} x=0$

Thus, $\lim\limits_{x\to 0}\dfrac{1-\cos x}{x}=\dfrac{1}{2}(1)^2(0)=0$

**Example 4 :** Show that $\lim\limits_{x\to 0}\dfrac{1-\cos 2x}{3\tan^2 x}=\dfrac{2}{3}$

**Solution :** We recall that $1-\cos 2x=2\sin^2 x$

Thus, $\lim\limits_{x\to 0}\dfrac{1-\cos 2x}{3\tan^2 x}=\lim\limits_{x\to 0}\dfrac{2\sin^2 x}{3\tan^2 x}$

$$=\lim_{x\to 0}\left(\frac{2}{3}\cos^2 x\right)$$

$$=\frac{2}{3}\left[\lim_{x\to 0}\cos x\right]^2$$

$$=\frac{2}{3}(1)^2=\frac{2}{3}$$

Hence, $\lim\limits_{x\to 0}\dfrac{1-\cos 2x}{3\tan^2 x}=\dfrac{2}{3}$

## EXERCISE 25.1

Evaluate the following limits :

1. $\lim\limits_{x\to 0}\dfrac{\sin 3x}{x}$

2. $\lim\limits_{x \to 0} \dfrac{\sin \frac{x}{4}}{2x}$

3. $\lim\limits_{x \to 0} \dfrac{x}{\tan x}$

4. $\lim\limits_{x \to 0} \dfrac{\sin x^2}{x}$

5. $\lim\limits_{x \to 0} \left( \dfrac{2 \sin x}{\tan x} + 5 \right)$

6. $\lim\limits_{x \to 0} \dfrac{\sin (2+x) - \sin (2-x)}{x}$

7. $\lim\limits_{t \to 0} \dfrac{\sin 4t}{\sin 2t}$

8. $\lim\limits_{x \to 0} \dfrac{\tan 5x}{\tan 7x}$

9. $\lim\limits_{t \to 0} \dfrac{1 - \cos 6t}{t}$

10. $\lim\limits_{t \to 0} \dfrac{1 - \cos 2t}{\sin 2t}$

11. $\lim\limits_{t \to \frac{\pi}{2}} \dfrac{\cos t}{t - \frac{\pi}{2}}$

$\left[ \textbf{Hint:} \text{ If } t \to \frac{\pi}{2}, \text{ then } t - \frac{\pi}{2} \to 0. \text{ Also, } \cos t = \sin \left( \frac{\pi}{2} - t \right) \right]$

12. $\lim\limits_{t \to \frac{\pi}{2}} \dfrac{\cot t}{\frac{\pi}{2} - t}$

*13. Prove that $\lim\limits_{y \to 1} \dfrac{1 - \frac{1}{y}}{\sin [\pi(y-1)]} = \dfrac{1}{\pi}$

## 25.3 Derivatives of sin x and cos x

We are now equipped to find the derivatives of sin $x$ and cos $x$. We will use the **delta method** to find the derivative of sin $x$. We let

$$f(x)=\sin x \tag{1}$$

Thus, $f(x+\triangle x)=\sin(x+\triangle x)$

*i.e.,*

$$f(x+\triangle x)=\sin x\cos\triangle x+\cos x\sin\triangle x \tag{2}$$

Subtracting (1) from (2) and dividing both sides by $\triangle x$, we have

$$\frac{f(x+\triangle x)-f(x)}{\triangle x}=\frac{\sin x(\cos\triangle x-1)+\cos x\sin\triangle x}{\triangle x} \tag{3}$$

Taking limits of both sides in (3), we have

$$\lim_{\triangle x\to 0}\frac{f(x+\triangle x)-f(x)}{\triangle x}=\lim_{\triangle x\to 0}\frac{\sin x(\cos\triangle x-1)+\cos x\sin\triangle x}{\triangle x}$$

$$=\lim_{\triangle x\to 0}\frac{\sin x(\cos\triangle x-1)}{\triangle x}+\lim_{\triangle x\to 0}\frac{\cos x\sin\triangle x}{\triangle x}$$

*i.e.,*

$$\lim_{\triangle x\to 0}\frac{f(x+\triangle x)-f(x)}{\triangle x}=\sin x\lim_{\triangle x\to 0}\frac{\cos\triangle x-1}{\triangle x}$$

$$+\cos x\lim_{\triangle x\to 0}\frac{\sin\triangle x}{\triangle x} \tag{4}$$

Now $\lim\limits_{\triangle x\to 0}\dfrac{\sin\triangle x}{\triangle x}=1$. We recall from Example 3, Section 25.2,

that,

$$\lim_{x\to 0}\frac{1-\cos x}{x}=0$$

Thus, $\lim\limits_{\triangle x\to 0}\dfrac{\cos\triangle x-1}{\triangle x}=0$

(4), therefore, becomes

$$\lim_{\triangle x\to 0}\frac{f(x+\triangle x)-f(x)}{\triangle x}=\sin x\times 0+\cos x\times 1$$

$$=\cos x$$

We have thus proved that

**Theorem 19 :** If $f(x)=\sin x,$

then. $f'(x)=\cos x$

In other words, the derived function of $\sin x$ is $\cos x$.

We will now find the derivative of $\cos x$. Rather than use the delta method, we will proceed as follows :

We let $g(x)=\cos x=\sin\left(\frac{\pi}{2}-x\right)$

Using **Chain Rule,** therefore, we have

$$g'(x)=\left[\cos\left(\frac{\pi}{2}-x\right)\right]\frac{d}{dx}\left(\frac{\pi}{2}-x\right) \tag{5}$$

But $\cos\left(\frac{\pi}{2}-x\right)=\sin x$. Thus (5), becomes

$$g'(x)=(\sin x)\times(-1)=-\sin x$$

We have thus proved that

**Theorem 20 :** If $g(x)=\cos x,$

then, $g'(x)=-\sin x,$

[The reader is advised to furnish a proof of Theorem 20, by using the delta method.]

**Remark :** We see that $\sin x$ and $\cos x$ are differentiable. It follows, therefore, from Theorem 15, that both $\sin x$ and $\cos x$ are continuous.

We now consider some examples.

**Example 1 :** Find $\frac{dy}{dx}$ if $y=(\cos x)^3$

**Solution :** We let $t=\cos x$ and use the **Chain Rule.** We have

$$y=t^3$$

Thus, $$\frac{dy}{dt}=3t^2$$

Also, since $t=\cos x$,

$$\frac{dt}{dx}=-\sin x$$

Now the **Chain Rule** tells us that

$$\frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx}$$

Thus, $$\frac{dy}{dx}=3t^2(-\sin x)$$

*i.e.*, $$\frac{dy}{dx}=-3\cos^2 x\sin x$$

**Example 2 :** Find the derivative of

$$f(t)=\sin\sqrt{t},\ t>0$$

with respect to $t$.

**Solution :** Using the **Chain Rule**, we have

$$f'(t)=\cos\left(\sqrt{t}\right)\frac{d}{dt}\left(\sqrt{t}\right)$$

$$=\cos\left(\sqrt{t}\right)\left(\frac{1}{2\sqrt{t}}\right)$$

Thus, $$f'(t)=\frac{1}{2\sqrt{t}}\cos\left(\sqrt{t}\right)$$

***Example 3 :** Find the derivative of

$$y=\frac{(\sin x^2)^3}{\sqrt{1+x^2}}$$

with respect to $x$. Hence, find the value of the derivative at $x=0$ and $x=\sqrt{\frac{\pi}{2}}$

**Solution :** We are given that

$$y=\frac{(\sin x^2)^3}{\sqrt{1+x^2}}$$

To find $\frac{dy}{dx}$, we use the **Quotient Rule** and get

$$\frac{dy}{dx}=\frac{\sqrt{1+x^2}\frac{d}{dx}\,[(\sin x^2)^3]-(\sin x^2)^3\frac{d}{dx}\,(\sqrt{1+x^2})}{(\sqrt{1+x^2})^2} \tag{1}$$

Now $\frac{d}{dx}\,[(\sin x^2)^3]=3(\sin x^2)^2\,\frac{d}{dx}\,(\sin x^2)$

$$=3(\sin x^2)^2\,(\cos x^2)\frac{d}{dx}\,(x^2)$$

$$=6x(\sin x^2)^2\,(\cos x^2)$$

And $\frac{d}{dx}\,(\sqrt{1+x^2})=\frac{1}{2}\,(1+x^2)^{-\frac{1}{2}}\,\frac{d}{dx}(1+x^2)$

$$=\frac{x}{\sqrt{1+x^2}}$$

Thus (1) becomes

$$\frac{dy}{dx}=\frac{\sqrt{1+x^2}\,[6x(\sin x^2)^2\,(\cos x^2)]-\dfrac{(\sin x^2)^3\,x}{\sqrt{1+x^2}}}{1+x^2}$$

*i.e.,* $$\frac{dy}{dx}=\frac{x(\sin x^2)^2\,[6(\cos x^2)(1+x^2)-\sin x^2]}{(1+x^2)^{\frac{3}{2}}} \tag{2}$$

To find the value of $\frac{dy}{dx}$ at $x=0$, we substitute $x=0$ in (2).

We get

$$\left.\frac{dy}{dx}\right]_{x=0}=0$$

Similarly, we get

$$\left.\frac{dy}{dx}\right]_{x=\sqrt{\frac{\pi}{2}}}=\frac{\sqrt{\frac{\pi}{2}}\left(\sin\frac{\pi}{2}\right)^2\left[6\left(1+\frac{\pi}{2}\right)\cos\frac{\pi}{2}-\sin\frac{\pi}{2}\right]}{\left(1+\frac{\pi}{2}\right)^{\frac{3}{2}}} \tag{3}$$

But $\sin\frac{\pi}{2}=1$ and $\cos\frac{\pi}{2}=0$. Thus, we have

$$\left.\frac{dy}{dx}\right]_{x=\sqrt{\frac{\pi}{2}}}=\frac{\sqrt{\frac{\pi}{2}}(-1)}{\left(\frac{2+\pi}{2}\right)^{\frac{3}{2}}}=-\frac{2\sqrt{\pi}}{(2+\pi)^{\frac{3}{2}}}$$

***Example 4 :** Find the derivative of

$$y=\sin x^2$$

with respect to $x$ using the delta method. You may use the following **important property of limits of composite functions :**

**Let f(x) be a function and c a real number.**

**Let,** $$\lim_{x\to c} f(x)=l$$

**Finally, let g be a function defined on an open interval containing $l$.**

**If,** $$\lim_{y\to l} g(y)=m,$$

**then,** $$\lim_{x\to c} g[f(x)]=m$$

**Solution :** We have

$$y=\sin x^2 \tag{1}$$

Thus, $$y+\Delta y=\sin (x+\Delta x)^2 \tag{2}$$

Subtracting (1) from (2) and dividing both sides by $\Delta x$, we get

$$\frac{\Delta y}{\Delta x}=\frac{\sin (x+\Delta x)^2-\sin x^2}{\Delta x} \tag{3}$$

We recall that

$$\sin A-\sin B=2 \sin \frac{A-B}{2} \cos \frac{A+B}{2}$$

Thus, $\sin (x+\Delta x)^2-\sin x^2=2 \sin\left(x\, \Delta x+\frac{(\Delta x)^2}{2}\right) \cos\left(x^2+x\, \Delta x+\frac{(\Delta x)^2}{2}\right)$

(3), therefore, becomes

$$\frac{\Delta y}{\Delta x}=\frac{2 \sin\left(x\, \Delta x+\frac{(\Delta x)^2}{2}\right) \cos\left(x^2+x\, \Delta x+\frac{(\Delta x)^2}{2}\right)}{\Delta x} \tag{4}$$

In (4), we take limits of both sides as $\Delta x\to 0$. We get

$$\frac{dy}{dx}=2\left[\lim_{\Delta x\to 0}\frac{\sin\left(x\,\Delta x+\frac{(\Delta x)^2}{2}\right)}{\Delta x}\right]\times\left[\lim_{\Delta x\to 0}\cos\left(x^2+x\,\Delta x+\frac{(\Delta x)^2}{2}\right)\right] \tag{5}$$

We need to evaluate the limits in the R.H.S. of (5). Let us do so.

$$\lim_{\Delta x\to 0}\frac{\sin\left(x\,\Delta x+\frac{(\Delta x)^2}{2}\right)}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{\sin(x\,\Delta x)\cos\left(\frac{(\Delta x)^2}{2}\right)+\cos(x\,\Delta x)\sin\left(\frac{(\Delta x)^2}{2}\right)}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{\sin(x\,\Delta x)\cos\left(\frac{(\Delta x)^2}{2}\right)}{\Delta x}+\lim_{\Delta x\to 0}\frac{\cos(x\,\Delta x)\sin\left(\frac{(\Delta x)^2}{2}\right)}{\Delta x}$$

*i.e.*, $$\lim_{\Delta x\to 0}\frac{\sin\left(x\,\Delta x+\frac{(\Delta x)^2}{2}\right)}{\Delta x}=\lim_{\Delta x\to 0}\frac{\sin(x\,\Delta x)}{\Delta x}\lim_{\Delta x\to 0}\cos\left(\frac{(\Delta x)^2}{2}\right)+\lim_{\Delta x\to 0}\cos(x\,\Delta x)\lim_{\Delta x\to 0}\frac{\sin\left(\frac{(\Delta x)^2}{2}\right)}{\Delta x} \tag{6}$$

Now as $\Delta x\to 0$, $(\Delta x)^2\to 0$ and hence $\frac{(\Delta x)^2}{2}\to 0$. Also as $\Delta x\to 0$, $x\Delta x\to 0$.

Thus,

$$\lim_{\Delta x\to 0}\frac{\sin(x\,\Delta x)}{\Delta x}=\lim_{\Delta x\to 0}\left(\frac{\sin(x\,\Delta x)}{x\,\Delta x}\cdot x\right)=x$$

$$\lim_{\Delta x\to 0}\cos\left(\frac{(\Delta x)^2}{2}\right)=1$$

$$\lim_{\Delta x\to 0}\cos(x\,\Delta x)=1$$

$$\lim_{\Delta x\to 0}\frac{\sin\left(\frac{(\Delta x)^2}{2}\right)}{\Delta x}=\lim_{\Delta x\to 0}\left[\frac{\sin\left(\frac{(\Delta x)^2}{2}\right)}{\frac{(\Delta x)^2}{2}}\quad\frac{\Delta x}{2}\right]$$

$$=\left[\lim_{\Delta x\to 0}\frac{\sin\left(\frac{(\Delta x)^2}{2}\right)}{\frac{(\Delta x)^2}{2}}\right]\left[\lim_{\Delta x\to 0}\frac{\Delta x}{2}\right]=0$$

Hence (6) becomes

$$\lim_{\Delta x\to 0}\frac{\sin\left(x\,\Delta x+\frac{(\Delta x)^2}{2}\right)}{\Delta x}=x(1)+1(0)=x \tag{7}$$

Next let us consider $\lim\limits_{\Delta x\to 0}\left[\cos\left(x^2+x\,\Delta x+\frac{(\Delta x)^2}{2}\right]\right.$

We use the property of limits of composite functions. We let

$$f(x)=x^2+x\,\Delta x+\frac{(\Delta x)^2}{2}$$

Then, $\lim\limits_{\Delta x\to 0} f(x)=x^2$

Also, $\lim\limits_{y\to x^2}\cos y=\cos x^2$, since $\cos y$ is continuous.

Hence $\lim\limits_{\Delta x\to 0}\cos[f(x)]=\cos x^2$

i.e., $\lim\limits_{\Delta x\to 0}\left[\cos\left(x^2+x\,\Delta x+\frac{(\Delta x)^2}{2}\right)\right]=\cos x^2$ (8)

Substituting (7) and (8) in (5), we get

$$\frac{dy}{dx}=2x\cos x^2$$

**Example 5 :** Find the equation of the tangent to the curve

$$y=\sin x$$

at $x=\frac{\pi}{4}$

**Solution :** We have $y=\sin x$

Thus, $$\frac{dy}{dx}=\cos x$$

The slope of the tangent at $x=\frac{\pi}{4}$, therefore, is $\left.\frac{dy}{dx}\right]_{x=\frac{\pi}{4}}=\cos\frac{\pi}{4}=\frac{1}{\sqrt{2}}$

Also, when $x=\frac{\pi}{4}$, $y=\sin\frac{\pi}{4}=\frac{1}{\sqrt{2}}$

The tangent at $x=\frac{\pi}{4}$, therefore, passes through $\left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right)$ and has a slope $\frac{1}{\sqrt{2}}$. Using the **point-slope form**, we can write its equation as

$$y-\frac{1}{\sqrt{2}}=\frac{1}{\sqrt{2}}\left(x-\frac{\pi}{4}\right)$$

Thus, $\sqrt{2}\,y-x+\frac{\pi-4}{4}=0$ is the required equation of the tangent.

**Example 6 :** Find the equation of the normal to

$$y=\cos(3t+4)$$

at $t=\frac{\pi-8}{6}$

**Solution :** Since $y=\cos(3t+4)$, it follows that

$$\frac{dy}{dt}=-3\sin(3t+4)$$

At $t=\frac{\pi-8}{6}$, we have $\frac{dy}{dt}=-3\sin\left(\frac{\pi}{2}\right)=-3$

The slope of the normal at $t=\frac{\pi-8}{6}$ is, therefore, $\frac{1}{3}$.

Also at $t=\frac{\pi-8}{6}$, $y=\cos\frac{\pi}{2}=0$

The normal at $t=\frac{\pi-8}{6}$, thus, passes through $\left(\frac{\pi-8}{6}, 0\right)$ and has a slope, $\frac{1}{3}$. Again, using the **point-slope form**, we write the equation of the normal as

$$y-0=\frac{1}{3}\left(t-\frac{\pi-8}{6}\right)$$

or, $$18y-6t+\pi-8=0,$$

which is the required equation

## EXERCISE 25.2

Use the **delta method** to find the derivative of each of the following :

1. $\cos x$
2. $\sin 2x$
3. $\cos 3x$

Find the derivative of each of the following :

4. $\sin 4x$
5. $\cos 5x$
6. $\sin (3x^{\frac{2}{3}}), x>0$
7. $\sqrt{\sin x^3}, 0<x<1$
8. $(x^2+\cos x)^4$
9. $x^3 \sin x+2x \cos x-2x \sin x$
10. $\frac{1}{a} \sin ax-\frac{1}{5b} \cos^5 bx$
11. $\cos (1-x^2)^2$
12. $\frac{\sin 2x}{x^3}$
13. $\sin 3x \sin^3 x$

**14.** $\sin^m x \sin mx$, $m$ is a positive integer.

***15.** Use the **delta method** to find $f'(x)$ if $f(x)=\cos(x^2+1)$

Find $f'$ for each of the following functions :

**16.** $f(x)=(3\sin^2 x-\cos^5 x+2)^3$

**17.** $f(t)=(t\sin t-\cos\sqrt{t})^5,\ t>0$

**18.** $f(u)=\dfrac{\sin^3 u\cos u}{\cos 3u}$

Evaluate the derivative of each of the following functions at indicated point(s):

**19.** $\dfrac{1-\sin x}{1+\cos x},\quad x=\dfrac{\pi}{2}$

**20.** $\cos\left(2x+\dfrac{\pi}{2}\right),\quad x=\dfrac{\pi}{3}$

**21.** $\dfrac{\sin^2 x+\cos x}{1+x^2},\quad x=0$ and $x=\dfrac{\pi}{2}$

**22.** $9\sin x+\sin 3x,\quad x=\dfrac{\pi}{3}$

**23.** $\sqrt{\sin^4 x^2+\cos^4 x^2},\quad x=0$ and $x=\dfrac{\sqrt{\pi}}{2}$

**24.** $\cos(\sin x^2),\quad x=\sqrt{\dfrac{\pi}{2}}$

**25.** $\dfrac{x^5\sin x(1+\cos x)}{\cos x},\quad x=0$ and $x=\dfrac{\pi}{4}$

Find the equation of the tangent to each of the following at the indicated point .

**26.** $y=\cos x$ at $x=\dfrac{\pi}{3}$

**27.** $y=2\sin x+\sin 2x$ at $x=\dfrac{\pi}{3}$

**28.** $y=4+\cos^2 x$ at $x=\dfrac{\pi}{2}$

29. $y=x-\sin x\cos x$ at $x=\frac{\pi}{2}$

30. $y=\frac{1+\sin x}{\cos x}$ at $x=\frac{\pi}{4}$

31. $y=\cos x\sin 2x-\sin x\cos 2x$ at $x=\frac{\pi}{3}$

Find the equation of the normal to each of the following at the indicated point :

32. $y=\sin^2 x$ at $x=\frac{\pi}{2}$

33. $y=\sin x+\cos x$ at $x=0$

34. $y=3\cos x+\cos 2x-5\sin x$ at $x=\frac{2\pi}{3}$

35. $y=2\sin^2 3x$ at $x=\frac{\pi}{6}$

36. $y=\frac{1+\sin x}{\cos x}$ at $x=\frac{\pi}{4}$

37. $y=x+\sin x\cos x$ at $x=\frac{\pi}{2}$

Determine whether the following functions are increasing or decreasing for stated values of $x$ :

38. $f(x)=x+\sin x$ for all $x$

39. $f(x)=x-\cos x$ for all $x$

40. $g(x)=-\frac{x}{2}+\sin x$ for $-\frac{\pi}{3}<x<\frac{\pi}{3}$

41. $g(x)=\cos\left(2x+\frac{\pi}{4}\right)$ for $\frac{3\pi}{8}<x<\frac{5\pi}{8}$

## 25.4 Differentiation of Other Trigonometric Functions

We now learn how to find the derivatives of tan $x$, cot $x$, sec $x$ and cosec $x$. *We recall that while sin $x$ and cos $x$ are defined for all real $x$, tan $x$, cot $x$, sec $x$ and cosec $x$ are not.* We will, however, not bother to state the domains of these functions every time we write them.

Let us first find the derivative of tan $x$. [Also, see Question 1, **Exercise 25.3**]

**We let**

$$f(x)=\tan x=\frac{\sin x}{\cos x}$$

We use the **Quotient Rule** and obtain

$$f'(x)=\frac{\cos x\left(\frac{d}{dx}\sin x\right)-\sin x\left(\frac{d}{dx}\cos x\right)}{(\cos x)^2}$$

$$=\frac{\cos^2 x+\sin^2 x}{\cos^2 x}=\frac{1}{\cos^2 x}=\sec^2 x$$

**We thus have**

**Theorem 21 : If $f(x)=\tan x$,**

**then, $f'(x)=\sec^2 x$**

In other words, the derived function of tan $x$ is $\sec^2 x$.

Next, we find the derivative of cot $x$. [Also, see Questions 2 and 5, **Exercise 25.3**]

**We let**

$$y=\cot x=\tan\left(\frac{\pi}{2}-x\right)$$

**We use the Chain Rule** and have

$$\frac{dy}{dx}=\sec^2\left(\frac{\pi}{2}-x\right)\frac{d}{dx}\left(\frac{\pi}{2}-x\right)$$

$$=-\sec^2\left(\frac{\pi}{2}-x\right)=-\operatorname{cosec}^2 x$$

**We thus have**

**Theorem 22 : If $f(x)=\cot x$**

**then, $f'(x)=-\operatorname{cosec}^2 x$**

[Alternatively, we can write $\cot x=\frac{\cos x}{\sin x}$ and use the **Quotient Rule** to find the derived function of cot $x$. The reader is advised to verify that he again gets the same derivative.]

Now we find the derivative of sec $x$. [Also, see Question 3, Exercise 25.3] We let

$$f(x)=\sec x=\frac{1}{\cos x}$$

We use the **Chain Rule** and have

$$f'(x)=-\frac{1}{(\cos x)^2}\frac{d}{dx}(\cos x)=\frac{+\sin x}{\cos^2 x}$$

$$=\frac{\sin x}{\cos x}\cdot\frac{1}{\cos x}=\tan x\sec x$$

We have thus proved the following important theorem :

**Theorem 23 :** If $f(x)=\sec x$,

then, $f'(x)=\sec x\tan x$

Finally, we find the derivative of cosec $x$. [Also, see Question 4, Exercise 25.3]

We let $y=\operatorname{cosec} x=\dfrac{1}{\sin x}$

We have $\dfrac{dy}{dx}=\dfrac{1}{(\sin x)^2}\cdot\cos x=-\dfrac{\cos x}{\sin x}\cdot\dfrac{1}{\sin x}$

$=-\operatorname{cosec} x\cot x$

We thus have

**Theorem 24 :** If $f(x)=\operatorname{cosec} x$,

then, $f'(x)=-\operatorname{cosec} x\cot x$

We make a table of **derivatives** of the six trigonometric functions that we have studied in Sections 25.3 and 25.4.

| f(x) | f′(x) |
|---|---|
| sin x | cos x |
| cos x | −sin x |
| tan x | $\sec^2 x$ |
| cot x | $-\operatorname{cosec}^2 x$ |
| sec x | sec x tan x |
| cosec x | −cosec x cot x |

We consider some examples. Again, we will not bother to state the **domains**. *We expect the student to find derivatives only at points where he can use the standard results, rather than the delta method.*

**Example 1 :** Find $f'(x)$ given that

$$f(x)=\sec^4 x$$

**Solution :** Since $f(x)=\sec^4 x$,

We have
$$f'(x)=4\sec^3 x\,\frac{d}{dx}(\sec x)$$
$$=4\sec^3 x\,(\sec x\tan x)$$

Thus,
$$f'(x)=4\sec^4 x\tan x$$

**Example 2 :** Find the derivative of

$$y=\cot^3 x-\tan\sqrt{x}$$

with respect to $x$.

**Solution :** We have

$$y=\cot^3 x-\tan\sqrt{x}$$

Thus
$$\frac{dy}{dx}=3\cot^2 x\left(\frac{d}{dx}\cot x\right)-(\sec^2\sqrt{x})\left(\frac{d}{dx}\sqrt{x}\right)$$
$$=-3\cot^2 x\,\mathrm{cosec}^2 x-\frac{1}{2\sqrt{x}}\sec^2\sqrt{x}$$

**Example 3 :** Differentiate $[\sin(3x+2)]^{\frac{7}{2}}$ with respect to $x$

**Solution :** We let

$$y=[\sin(3x+2)]^{\frac{7}{2}}$$

and,
$$t=\sin(3x+2) \qquad (1)$$
$$y=t^{\frac{7}{2}} \qquad (2)$$

Then,

We can now use the **Chain Rule**. We need $\frac{dy}{dt}$ and $\frac{dt}{dx}$.

From (1), we have

$$\frac{dt}{dx} = 3 \cos (3x+2)$$

From (2), we have

$$\frac{dy}{dx} = \frac{7}{2}\, t^{\frac{5}{2}}$$

Now. $$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

Thus, $$\frac{dy}{dx} = \frac{7}{2}\, t^{\frac{5}{2}} \left[ 3 \cos ( 3x+2 ) \right]$$

i. e., $$\frac{dy}{dx} = \frac{21}{2} \left[ \sin ( 3x+2 ) \right]^{\frac{5}{2}} \cos ( 3x+2 )$$

**Example 4** : Find the derivative of

$$y = ( \operatorname{cosec}^2 x + \tan x + \cot x )^7$$

and find $\left.\frac{dy}{dx}\right]_{x = \frac{\pi}{4}}$

**Solution** : Since $y = (\operatorname{cosec}^2 x + \tan x + \cot x )^7$,

we have $$\frac{dy}{dx} = 7 ( \operatorname{cosec}^2 x + \tan x + \cot x )^6 \frac{d}{dx} ( \operatorname{cosec}^2 x + \tan x + \cot x )$$

i. e., $$\frac{dy}{dx} = 7( \operatorname{cosec}^2 x+\tan x+ \cot x )^6( -2 \operatorname{cosec}^2 x \cot x+\sec^2 x-\operatorname{cosec}^2 x )$$

At $x = \frac{\pi}{4}$, $\tan \frac{\pi}{4} = \cot \frac{\pi}{4} = 1$, $\sec\frac{\pi}{4} = \sqrt{2} = \operatorname{cosec}\frac{\pi}{4}$

Thus, $$\left.\frac{dy}{dx}\right]_{x = \frac{\pi}{4}} = 7 ( 2+1+1 )^6 [ -2 (2) (1) + 2-2 ]$$

$$= -114688$$

**Example 5** : If $y^2 \sin x + y \tan x + (1+x^2) \cos x = 0$, find $\frac{dy}{dx}$.

**Solution** ; We differentiate implicitly and get

$$( y^2\cos x + 2y\frac{dy}{dx} \sin x ) + ( y \sec^2 x + \frac{dy}{dx} \tan x )$$

$$+ [ ( 1+x^2 ) ( -\sin x ) + 2x \cos x ] = 0$$

Thus, $\frac{dy}{dx} [ 2y \sin x + \tan x ] = ( 1+x^2 ) \sin x - ( y^2+2x ) \cos x - y \sec^2 x$

Hence, $\frac{dy}{dx} = \frac{(1+x^2) \sin x - (y^2+2x) \cos x - y \sec^2 x}{2y \sin x + \tan x}$

**Example 6** : If $y \sec x + \tan y + xy = 0$, prove that

$$(\sec x + \sec^2 y + x) \frac{dy}{dx} + y (\sec x + \tan x + 1) = 0$$

**Solution** : We are given

$$y \sec x + \tan y + xy = 0$$

We differentiate implicitly and get

$$( y \sec x \tan x + \frac{dy}{dx} \sec x ) + ( \sec^2 y) \frac{dy}{dx} + ( x\frac{dy}{dx} + y ) = 0$$

Hence, $\frac{dy}{dx} (\sec x + \sec^2 y + x) + y ( \sec x \tan x + 1 ) = 0$.

which proves the desired result.

**Example 7** : Find the equation of the tangent to

$$y = \tan^2 x - 2 \tan x + 2$$

at $x = \frac{\pi}{4}$.

**Solution** : At $x = \frac{\pi}{4}$, $y = \tan^2 \frac{\pi}{4} - 2 \tan \frac{\pi}{4} + 2 = 1$

We now find the derivative of $y$ with respect to $x$. We have

$$\frac{dy}{dx} = 2 \tan x \sec^2 x - 2 \sec^2 x$$

Thus, $\left.\frac{dy}{dx}\right]_{x=\frac{\pi}{4}} = 2(1)(\sqrt{2})^2 - 2(\sqrt{2})^2 = 0$

The slope of the tangent at $x = \frac{\pi}{4}$ is thus zero this means that we have a horizontal tangent at $x = \frac{\pi}{4}$, whose equation can easily be written as

$$y = 1$$

**Example 8** : Find equation of the normal to

$$y = 2 \sec x + \cot x$$

at $x = \frac{\pi}{4}$.

**Solution** : When $x = \frac{\pi}{4}$, $y = 2 \sec\frac{\pi}{4} + \cot\frac{\pi}{4} = 2\sqrt{2} + 1$

We now find $\frac{dy}{dx}$. We have

$$\frac{dy}{dx} = 2 \sec x \tan x - \operatorname{cosec}^2 x$$

Thus, $\left.\frac{dy}{dx}\right]_{x=\frac{\pi}{4}} = 2(\sqrt{2})(1) - (\sqrt{2})^2 = 2(\sqrt{2} - 1)$

The slope of the normal at $x = \frac{\pi}{4}$ is, therefore,

$$-\frac{1}{2(\sqrt{2} - 1)} = -\frac{\sqrt{2} + 1}{2}$$

Thus, the normal at $x = \frac{\pi}{4}$ passes through the point $\left(\frac{\pi}{4}, 2\sqrt{2} + 1\right)$ and has a slope $-\frac{\sqrt{2}+1}{2}$. Its equation, therefore, is

$$y-(2\sqrt{2}+1) = -\frac{\sqrt{2}+1}{2}\left[x-\frac{\pi}{4}\right] \qquad (1,$$

Multiplying both sides of (1) by $\frac{2}{\sqrt{2}+1} = 2\,(\sqrt{2}-1)$, we rewrite (1) as

$$2\,(\sqrt{2}-1)y-2\,(\sqrt{2}-1)\,(2\sqrt{2}+1)+x-\frac{\pi}{4}=0$$

Or, $2\,(\sqrt{2}-1)\,y+x-\left[6-2\sqrt{2}+\frac{\pi}{4}\right]=0$

which is the equation of the required normal.

## EXERCISE 25.3

Use the **delta method** to find the derivative of each of the following:

1. $\tan x$
2. $\cot x$
3. $\sec x$
4. $\operatorname{cosec} x$
5. Write $\cot x=\tan\left(\frac{\pi}{2}-x\right)$. Use **Chain Rule** to show that the derivative of $\cot x$ is $-\operatorname{cosec}^2 x$.

Find the derivative of each of the following :

6. $\cot^3 x^2$
7. $\tan x^{\frac{3}{2}}$
8. $\sqrt{\sec^2 x+\operatorname{cosec}^2 x}$
9. $\frac{1}{5}\tan x\sin 2x$
10. $(x\cot^3 x)^{\frac{3}{2}}$
11. $\dfrac{1+\tan x}{1-\tan x}$
12. $\dfrac{\sec x+\tan x}{\sec x-\tan x}$

13. $x^3 \tan^2 \frac{x}{2}$

14. $\sqrt{x^3 \sin x + \tan x}$

15. $(\tan^2 x + \operatorname{cosec}^2 x + 2)^5$

16. $(x^2 - 1)\left(\cot x + \frac{\tan x}{1+x^2}\right)$

17. $\tan (x^2 + x + 2)$

18. $\tan^3 (ax + b)$

19. $(x \tan x - x^3 \sin x + \sec x)^{\frac{3}{5}}$

20. $\left(\operatorname{cosec} x^2 - 3 \cos \sqrt{x} + \frac{1}{x^2} \cot x^3\right)^{\frac{1}{4}}$

21. $\left(\frac{2 \tan x}{\tan x + \cos x}\right)^2$

Evaluate the derivative of each of the following functions at the indicated point(s) :

22. $x^3 \tan x + 3 x^4 \sin x$, $x = 0$ and $x = \frac{\pi}{4}$

23. $\cot x + \sec^2 x + 5$, $x = \frac{\pi}{6}$

24. $(\operatorname{cosec} x + \sin x + \tan x)^3$, $x = \frac{\pi}{4}$

Find $\frac{dy}{dx}$ in each of the following :

25. $y \tan x - y^2 \cos x + 2x = 0$

26. $\sin^2 x + 2 \cos y + xy = 0$

27. $\sin (xy) + \frac{x}{y} = x^2 - y$

28. $(1 + y^2) \sec x - y \cot x + 1 = x^2$

29. $2y^3 + \frac{y}{1+x^3} + \tan^2 x + \sin y = 0$

30. $\tan(x+y)+\tan(x-y)=1$

31. $\cos(x+y)=y\sin x$

32. $\cot(xy)+xy=y$

Find equation of the tangent to each of the following at the indicated point :

33. $y=\tan x$ at $x=\frac{\pi}{4}$

34. $y=\sec^4 x-\tan^4 x$ at $x=\frac{\pi}{3}$

35. $y=\cot x^2+\tan^2 x^2+2$ at $x=\frac{\sqrt{\pi}}{2}$

36. $y=\sin^2 x+\cot^2 x+3$ at $x=\frac{9\pi}{4}$

Find equation of the normal to each of the following at the indicated point .

37. $y=\cot x$ at $x=\frac{\pi}{4}$

38. $y=\frac{\tan x+\sin 2x}{1+x^2}$ at $x=\pi$

39. $y=(\sin 2x+\cot x+2)^2$ at $x=\frac{\pi}{2}$

Determine whether the following functions are increasing or decreasing for stated values of $x$. :

40. $f(x)=\tan x-1$ for all $x\neq\frac{(2n+1)\pi}{2}$, $n$ an integer

41. $f(x)=\tan x-4x$ for $-\frac{\pi}{3}<x<0$

## 25.5 Graphs of sin x and cos x

We now learn how to draw graphs of $\sin x$ and $\cos x$. We have seen that both $\sin x$ and $\cos x$ are continuous for all real values of $x$.

We recall that

$$\sin (x+2\pi) = \sin x$$

and, $$\cos (x+2\pi) = \cos x$$

i.e., **both sin x and cos x are periodic with a period $2\pi$.**

*It suffices, therefore, to draw a graph of $y = \sin x$ or $y = \cos x$ in $0 \leqslant x \leqslant 2\pi$.*

**We first draw a graph of $y = \sin x$**

Its derivative is $\frac{dy}{dx} = \cos x$. We recall that $\cos x > 0$ in $0 < x < \frac{\pi}{2}$.

Thus, **sin x increases in $0 \leqslant x \leqslant \frac{\pi}{2}$.**

Again, we recall that $\cos x < 0$ in $\frac{\pi}{2} < x < \frac{3\pi}{2}$. **sin x, therefore, decreases** in $\frac{\pi}{2} \leqslant x \leqslant \frac{3\pi}{2}$.

By examining sign of $\cos x$ in the interval $\frac{3\pi}{2} < x < 2\pi$, we can similarly conclude that **sin x increases in $\frac{3\pi}{2} \leqslant x \leqslant 2\pi$.**

We construct a table of values of $x$ and $\sin x$. [In the same table, we also list the corresponding values of $\cos x$.]

| $x$ | $0$ | $\frac{\pi}{6}$ | $\frac{\pi}{4}$ | $\frac{\pi}{3}$ | $\frac{\pi}{2}$ | $\frac{3\pi}{4}$ | $\pi$ | $\frac{5\pi}{4}$ | $\frac{3\pi}{2}$ | $\frac{7\pi}{4}$ | $2\pi$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\sin x$ | $0$ | $\frac{1}{2}$ | $\frac{1}{\sqrt{2}}$ | $\frac{\sqrt{3}}{2}$ | $1$ | $\frac{1}{\sqrt{2}}$ | $0$ | $-\frac{1}{\sqrt{2}}$ | $-1$ | $-\frac{1}{\sqrt{2}}$ | $0$ |
| $\cos x$ | $1$ | $\frac{\sqrt{3}}{2}$ | $\frac{1}{\sqrt{2}}$ | $\frac{1}{2}$ | $0$ | $-\frac{1}{\sqrt{2}}$ | $-1$ | $-\frac{1}{\sqrt{2}}$ | $0$ | $\frac{1}{\sqrt{2}}$ | $1$ |

We note that $y = \sin x$ has horizontal tangents at $\frac{\pi}{2}$ and $\frac{3\pi}{2}$. (Why ?)

A graph of $y=\sin x$ for $0\leqslant x\leqslant 2\pi$ is given in Fig. 25.3.

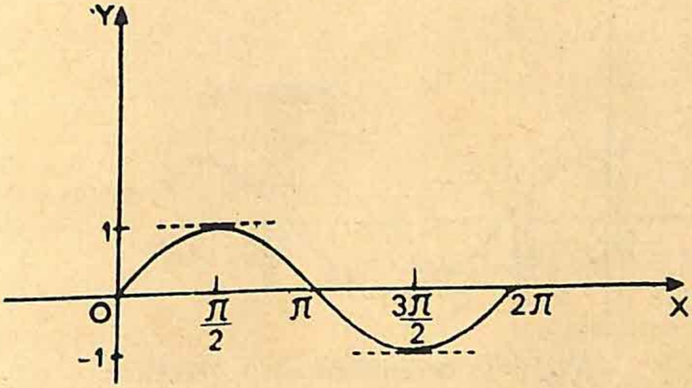

Fig. 25.3 : Graph of $y=\sin x$, $0\leqslant x\leqslant 2\pi$

Because of periodicity of sin **x**, it is possible to extend this graph on both sides

A graph of $y=\sin x$ for all real $x$ is shown in Fig. 25.4.

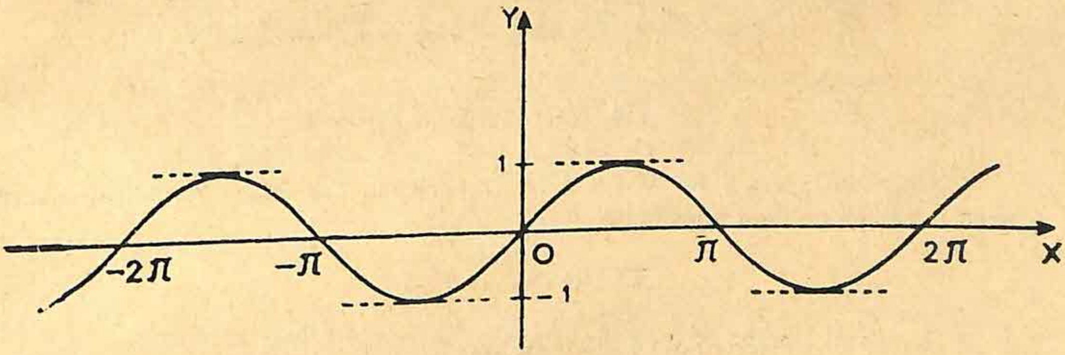

Fig. 25.4 : Graph of $y=\sin x$

By a similar analysis, the reader is advised to draw a graph of $y=\cos x$.

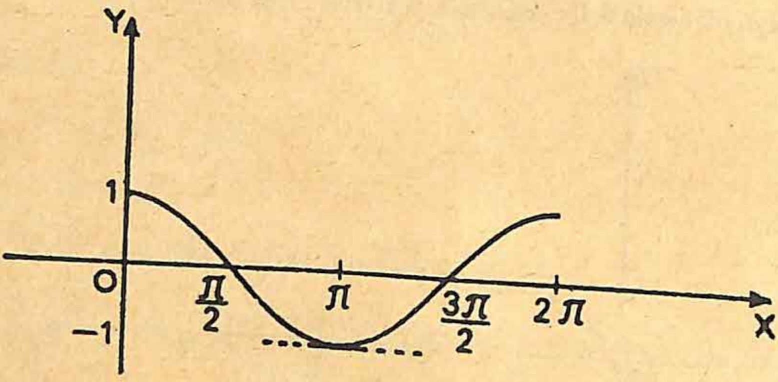

Fig. 25.5 : Graph of $y = \cos x$, $0 \leqslant x \leqslant 2\pi$

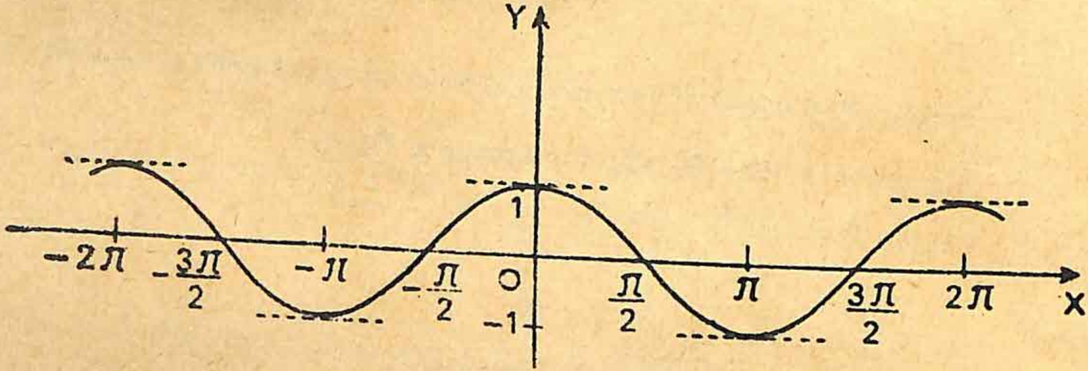

Fig. 25.6 : Graph of $y = \cos x$

Graph of $y = \cos x$ for $0 \leqslant x \leqslant 2\pi$ is shown in Fig. 25.5. Periodicity is used to extend the graph on both sides in Fig. 25.6.

## EXERCISE 25.4

1. Draw a graph of $y = 2 \sin x$, $0 \leqslant x \leqslant 2\pi$.
2. Recall the period of $y = \sin 2x$. Draw a graph of $y = \sin 2x$.
3. Draw graphs of $y = \sin x$ and $y = \sin 2x$ on the same axes.
4. Draw graphs of $y = \sin x$ and $y = -\sin x$ on the same axes.
5. Draw a graph of $y = 3 \cos x$, $0 \leqslant x \leqslant 2\pi$.

6. Recall the period of $y=\cos 2x$. Draw a graph of $y=-\cos 2x$.
7. Draw graphs of $y=\cos x$ and $y=\cos 2x$ on the same axes.
8. Draw graphs of $y=\cos x$ and $y=-\cos x$ on the same axes.

## 25.6 Key Concepts

$\lim\limits_{t\to 0} \frac{\sin t}{t}$ Graphs of sin x and cos x

Derivatives of trigonometric functions

## 25.7 Suggestions for Further Reading

*Any standard text on Calculus can be consulted for further reading, for instance,*

[1] of Section 18.7, [1] of Section 20.5, [1] of Section 21.9, [1] of Section 24.8.

*A proof of* $\lim\limits_{t\to 0} \frac{\sin t}{t} = 1$ *using the perimeter of a regular n-gon inscribed in a circle is given in*

[1] M.S. Knebelman : An Elementary Limit.

American Mathematical Monthly (Journal), Vol. 50 (1943), page 567.

*An alternative and elegant proof of* $\lim\limits_{t\to 0} \cos t = 1$ *is given in*

[2] M.J. Pascual : **On the** $\lim\limits_{\theta\to 0} \cos \theta$

American Mathematical Monthly (Journal), Vol. 62 (1955), pages 252-253

*Still another proof of* $\lim\limits_{t\to 0} \frac{\sin t}{t} = 1$ *is given in*

[3] S. Hoffman : A classroom Proof of $\lim\limits_{t\to 0} \frac{\sin t}{t} = 1$.

American Mathematical Monthly (Journal), Vol. 67 (1960), pages 671—672

## UNIT XXVI

# MAXIMA AND MINIMA

*The scientists and mathematicians of the 17th century were concerned with another class of problems, besides the* **problem of speed** and the **problem of tangents**, *pertaining to trajectories of cannon balls and other kinds of projectile paths as also the motion of heavenly bodies which essentially led to finding the maxima and minima of functions. In this unit, we study techniques for determining the maxima and minima of functions.*

## 26.1 HOW HIGH WILL THE BALL GO ?

*A ball is thrown vertically upwards with an initial speed of 29.4 m/sec. The height s, above the ground, at any instant t is given by*

$$s = 29.4t - 4.9t^2$$

*How high will the ball go ?* To answer this question, let us write

$$s = -4.9\,(t^2 - 6t) = -4.9\,[\,(t-3)^2 - 9\,]$$

i. e., $s = -4.9\,(t-3)^2 + 44.1$ (1)

To find the maximum value of $s$, it is clear that we should try to make the contribution of the term involving $t$ in (1) as small as possible. (Why ?) This, of course, necessitates that $t = 3$.

**Thus the maximum height that the ball can attain is 44.1 metres and this, the ball attains at the end of 3 seconds.**

We ask another natural question. *What will be the velocity of the ball at the instant when it reaches its maximum height ?*

We compute $\left.\dfrac{ds}{dt}\right]_{t=3}$. We have $\dfrac{ds}{dt} = 29.4 - 9.8t$ (2)

Thus.

$$\left.\frac{ds}{dt}\right]_{t=3} = 29.4-9.8t\Big]_{t=3} = 0$$

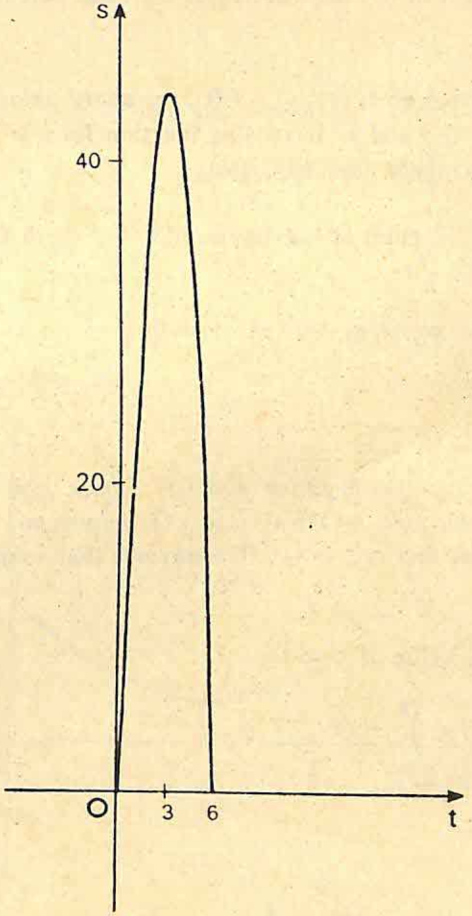

Fig. 26.1 : Graph of $s = 29.4t-4.9t^2$

We see that **the velocity at the precise instant when the ball reaches its maximum height is zero.**

Next we *examine the velocity before the ball attains its maximum height and also after it has reached its maximum height.* We see from (2) that

$$\frac{ds}{dt} = -9.8\,(t-3)$$

Thus, when $t < 3$, $t-3 < 0$ and, therefore, $\frac{ds}{dt} > 0$. Also, when $t > 3$, $t-3 > 0$ and, therefore, $\frac{ds}{dt} < 0$. This tells us that the ball is going higher and higher for $t < 3$ and that for $t > 3$, the height of the ball decreases i. e., the ball starts falling back to the ground.

**If we plot $s$ as a function of t, say, $s = f(t)$, the above information tells us that s is an increasing function for $t \leqslant 3$ and a decreasing function for $t \geqslant 3$. Also, that for $t = 3$, the graph has a horizontal tangent. (See Fig. 26.1)**

We say that $t = 3$ is the **point of maximum** of $s = f(t)$ in the sense that *s has the maximum value for this value of t.*

Let us consider another problem.

## 26.2 The Number Game

*How should we choose two non-negative numbers whose* **sum** *is $\frac{1}{2}$ so that the sum of the square of the first and the cube of the second is the minimum*? Let us assume that the second number is $x$. Then the first is $\frac{1}{2} - x$. **It is obvious that x can take values between 0 and $\frac{1}{2}$.**

We have to choose the value of $x$ so that

$$\left(\frac{1}{2} - x\right)^2 + x^3$$

has the minimum value.

We let

$$y = \left(\frac{1}{2} - x\right)^2 + x^3$$

i. e.,

$$y = \frac{1}{4} - x + x^2 + x^3$$

We see that the situation here is very much different from the previous example. It is not easy to express $y$ in a form that will help us to find $x$ for which $p$ has the minimum value. We feel, however, that if, at all, we can determine intervals where y increases decreases, we might be slightly better off. Calculus helps us in doing so. We have

$$\frac{dy}{dx} = -1 + 2x + 3x^2$$

$$= 3\left(x - \frac{1}{3}\right)(x+1)$$

Now $x+1$ is positive. (Why ?) Thus the sign of $\frac{dy}{dx}$ is determined by the sign of $x - \frac{1}{3}$. We see that $\frac{dy}{dx} < 0$ if $x < \frac{1}{3}$ and $\frac{dy}{dx} > 0$ if $x > \frac{1}{3}$. Also that $\frac{dy}{dx} = 0$ if $x = \frac{1}{3}$.

Thus $y$ decreases for $x \leqslant \frac{1}{3}$ and increases for $x \geqslant \frac{1}{3}$. At $x = \frac{1}{3}$, therefore, $y$ has the minimum value. (See Fig. 26.2)

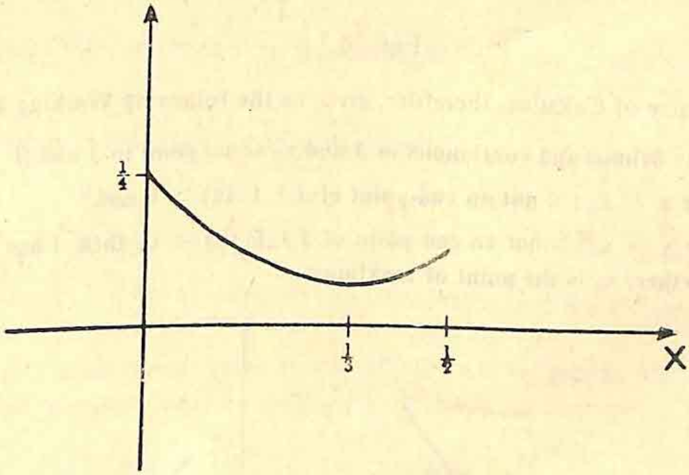

Fig 26.2 : Graph of $y = \left(\frac{1}{2} - x\right)^2 + x^3,\ 0 \leqslant x \leqslant \frac{1}{2}$

We say that $x = \frac{1}{3}$ is the **point of minimum** of $y = \left(\frac{1}{2} - x\right)^2 + x^3,\ 0 \leqslant x \leqslant \frac{1}{2}$, in the sense that *y has the minimum value for this value of x.*

[ Can you write the two numbers now ? They are $\frac{1}{6}$ and $\frac{1}{3}$. ]

## 26.3 A Working Rule to Find The Maximum and The Minimum of Certain Functions

By now it should be intuitively clear that if $f$ is increasing on the left of some point $x_0$ and decreasing on its right, then it has the maximum value at the point $x_0$. ( See Fig. 26.3 )

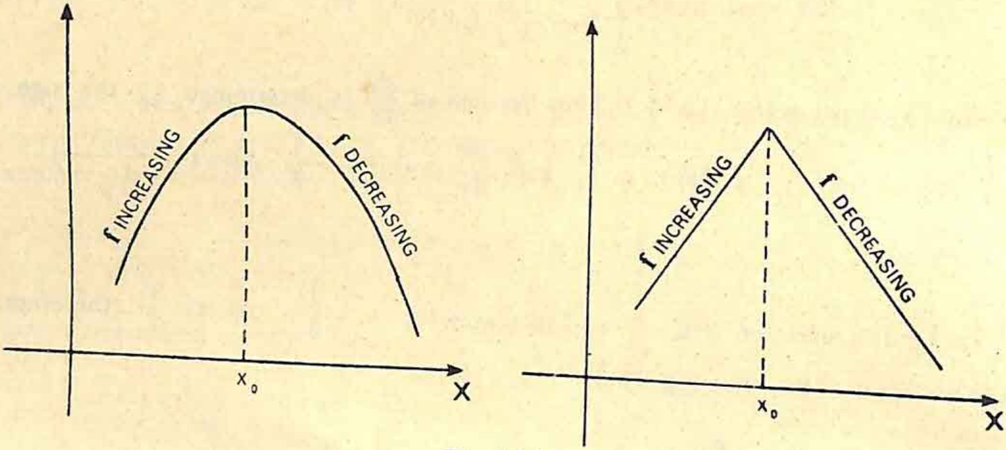

Fig. 26.3

Our knowledge of Calculus, therefore, gives us the following **Working Rule, WR1.**

**WR1 : If f is defined and continuous in J and $x_0$ some point in J and if**

**(i) for $x < x_0$ ( x not an end-point of J ), $f'(x) > 0$ and**

**(ii) for $x > x_0$ ( x not an end-point of J ), $f'(x) < 0$, then f has the maximum value at $x_0$. Further, $x_0$ is the point of maximum.**

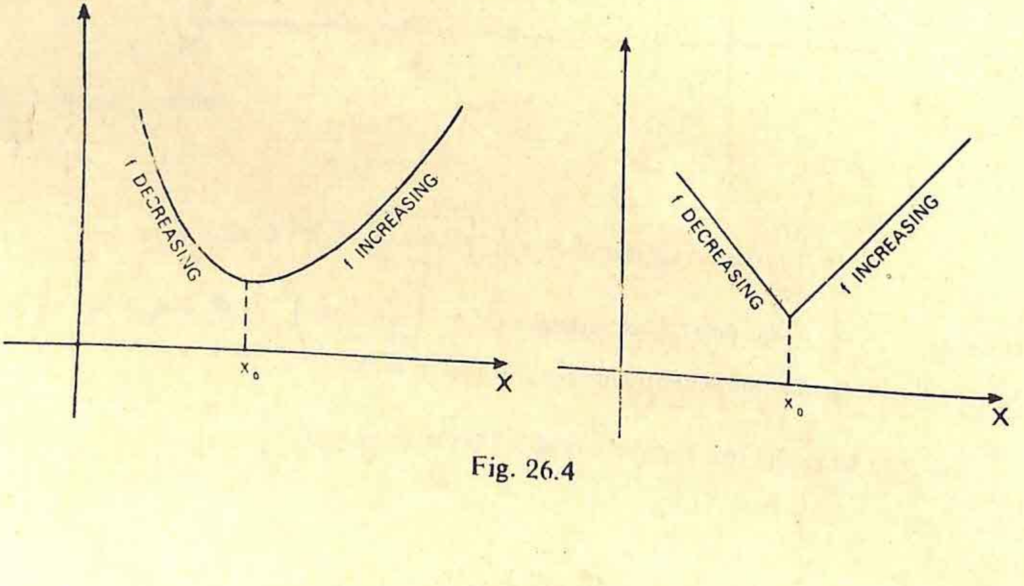

Fig. 26.4

What if we were looking for the minimum value at $x_0$ ? Again, it is intuitively clear that if $f$ is decreasing on the left of some point $x_0$ and increasing on its right, then it has the minimum value at the point $x_0$. ( See Fig. 26.4 ) Thus we have the following **Working Rule, WR2.**

**WR2 : If f is defined and continuous in J and $x_0$ some point in J and if**

**(i) for $x < x_0$ ( x not an end-point of J ), $f'(x) < 0$ and**

**(ii) for $x > x_0$ ( x not an end-point of J ), $f'(x) > 0$, then f has the minimum value at $x_0$. Further, $x_0$ is the point of minimum.**

We consider some examples.

**Example 1** : The **Revenue Function R (x)**, in rupees, of selling $x$ items* ( $x \geqslant 0$ ) of a certain manufacturing concern is

$$R(x) = 4000x - 2x^2$$

Determine $x$ at which the concern makes its maximum revenue. Find also the maximum revenue.

**Solution** : Clearly $R(x)$ is continuous. We have

$$R'(x) = 4000 - 4x$$

i. e., $$R'(x) = 4(1000-x)$$

Now, whenever $x < 1000$, $R'(x) > 0$ and whenever $x > 1000$, $R'(x) < 0$.

Thus $R(x)$ has the maximum value at $x = 1000$. In other words, *the manufacturing concern makes the maximum revenue by selling 1000 items.*

$$\text{Maximum Revenue} = R(x)\Big]_{x=1000} = 2000000.$$

Thus, *the maximum revenue which the concern makes is Rs 2000000.*

**Example 2** : Determine the minimum value of

$$f(x) = \sin x$$

in $\pi \leqslant x \leqslant 2\pi$.

---

*Although, as an economic variable, x would take positive integral values, we let x take values in the interval $x \geqslant 0$, while viewing it as a mathematical problem.

**Solution :** Clearly $f(x)$ is continuous. We have

$$f'(x) = \cos x$$

Now, whenever $\pi \leqslant x < \frac{3\pi}{2}$, $f'(x) < 0$ and whenever $\frac{3\pi}{2} < x \leqslant 2\pi$, $f'(x) > 0$.

Thus, $f(x) = \sin x$ attains the minimum value at $x = \frac{3\pi}{2}$.

$$\text{Minimum value} = \sin x \Big]_{x = \frac{3\pi}{2}} = -1$$

## EXERCISE 26.1

1. An object is thrown vertically upwards, on the surface of the moon, with an initial speed of 6.4m/sec. The height s, above the surface of the moon, at any instant t is given by

   $$s = 6.4t - 0.8t^2.$$

   How high will the object go ? How much time will it take to reach the maximum height ?

2. A manufacturer can sell $x$ items ($x \geqslant 0$) at a price of Rs $(330 - x)$ each. The cost of producing $x$ items is Rs $x^2 + 10x + 12$. Determine the number of items to be sold so that the manufacturer can make the maximum profit.

   [**Hint** : $P(x) = x(330 - x) - (x^2 + 10x + 12)$]

3. Determine the maximum value of

   $$f(x) = \sin 2x$$

   in $0 \leqslant x \leqslant \frac{\pi}{2}$

4. Find the point of maximum of

   $$y = 4 + 3x - 2x^2$$

   in $0 \leqslant x \leqslant 1$.

5. Find the minimum value of

$$y=x^3-3x$$

in $0\leqslant x\leqslant 2$

6. Determine the minimum value of

$$y=\left| x-\frac{1}{2} \right|$$

7. Amongst all pairs of positive numbers with product 64, find those whose sum is the least.

8. The **Profit Function,** in rupees, of a firm selling $x$ $(x\geqslant 0)$ items per week is given by

$$P(x)+3500=(400-x)x$$

How many items should the firm sell to make the maximum profit ? Determine also the maximum profit.

9. A beam of length $l$ is supported at one end. If w is the uniform load per unit length, the **bending moment** M at a distance $x$ from the end is given by

$$M=\frac{1}{2}\,l\,x-\frac{1}{2}\,wx^2$$

Find the point on the beam at which the bending moment has the maximum value.

10. Find the point of minimum and the minimum value of

$$f(x)=\frac{1}{3}\,x^3+\frac{1}{2}\,x^2-2x+11$$

for $x>0$.

*11 Determine the point of maximum of

$$f(x)=\sin x+\cos x$$

in $0\leqslant x\leqslant \frac{\pi}{2}$.

$\left[\textbf{Hint}: \cos x - \sin x = \sqrt{2} \cos\left(\frac{\pi}{4} + x\right)\right]$

*12. Of all the closed cylindrical tin cans (right circular) which enclose a given volume of 100 cubic cm, which has the minimum surface area ?

[**Hint** : Volume $V = \pi r^2 h$ and Surface Area $S = 2\pi r^2 + 2\pi r h$.

Express $h$ as $\frac{V}{\pi r^2} = \frac{100}{\pi r^2}$. Thus $S(r) = 2\pi r^2 + \frac{200}{r}$. Find $S'(r)$.]

## 26.4 The Number Game Again

An important question now poses itself. **Do the Working Rules that we have developed in Section 26.3 work for all kinds of functions ?** The answer to this question is "**No**", as will be clear from the following example :

In the Number Game example of Section 26.2, we modify the restriction on numbers slightly and ask. "*How should we choose two numbers, each less than or equal to 2, whose sum is* $\frac{1}{2}$ *so that the sum of the square of the first and the cube of the second is the minimum ?*"

As before, we let the second number be $x$ and the first number be $\frac{1}{2} - x$.

Since both $x \leqslant 2$ and $\frac{1}{2} - x \leqslant 2$, **it is clear that x can take values** $-\frac{3}{2} \leqslant x \leqslant 2$.

We have to choose the value of $x$ so that

$$y = f(x) = \left(\frac{1}{2} - x\right)^2 + x^3$$

i.e., $y = \frac{1}{4} - x + x^2 + x^3$

has the minimum value.

As before, we find $\frac{dy}{dx}$. We have

$$\frac{dy}{dx} = -1 + 2x + 3x^2 = 3\left(x - \frac{1}{3}\right)(x+1)$$

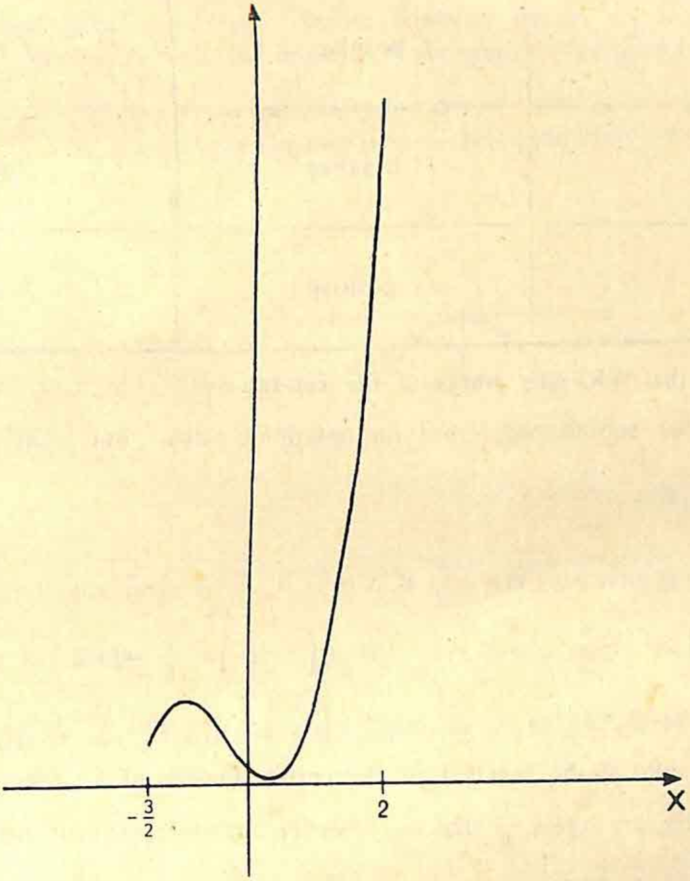

Fig. 26.5 : Graph of $y = \left(\frac{1}{2} - x\right)^2 + x^3, -\frac{3}{2} \leqslant x \leqslant 2$

Clearly $\frac{dy}{dx}$ is positive if both $\left(x - \frac{1}{3}\right)$ and $(x+1)$ are positive or both negative, i.e., if $x > \frac{1}{3}$ or $x < -1$. And $\frac{dy}{dx}$ is negative if $\left(x - \frac{1}{3}\right)$ and $(x+1)$ are of opposite signs, i.e., if $-1 < x < \frac{1}{3}$. Thus we have the following situation :

| | | |
|---|---|---|
| $-\frac{3}{2} \leqslant x < -1$ | $f'$ positive | $f$ increasing |
| $-1 < x < \frac{1}{3}$ | $f'$ negative | $f$ decreasing |
| $\frac{1}{3} < x \leqslant 2$ | $f'$ positive | $f$ increasing |

We see that **WR2 only works in the sub-interval** $-1 < x \leqslant 2$ and tells us that **at** $x = \frac{1}{3}$, in this sub-interval, $f$ has the minimum value. But what about the sub-**interval** $-\frac{3}{2} \leqslant x < -1$ ?

Since $f$ is increasing in $-\frac{3}{2} \leqslant x < -1$, it is clear that $f$ takes values larger than $f\left(-\frac{3}{2}\right)$ for $-\frac{3}{2} < x < -1$. But $f\left(-\frac{3}{2}\right) = \frac{5}{8}$ which, of course, is larger than the function-value at $x_0 = \frac{1}{3}$ $\left[\text{Since } f\left(\frac{1}{3}\right) = \frac{7}{108} \text{ and } \frac{5}{8} > \frac{7}{108}\right]$. **Thus f** $\left(\frac{1}{3}\right)$ **is the minimum value of the function in the entire domain of f.** (See Fig. 26.5) The required numbers are again $\frac{1}{6}$ and $\frac{1}{3}$. We see, therefore, that *although* $x_0 = \frac{1}{3}$ *is the point of minimum of* $f$ *in the entire domain* $-\frac{3}{2} \leqslant x \leqslant 2$, *we could not determine this point of minimum by WR2.* **We, therefore, need an alternative criterion to be able to find the maximum and minimum value of such functions.** How shall we develop this alternative criterion ?

We recall from the examples above that **if there is a tangent at the point of maximum or minimum, it is horizontal, i.e., it has slope zero.** Can we use this as a criterion or as a motivation for developing a criterion ? We ask, therefore, **that must this be true for such points (of maximum or minimum) for all functions ?** The answer is that indeed **it is true for a great variety of functions if there is a tangent at such points.**

Rather than give an analytic proof, we again appeal to our intuition and make an argument which we will also find useful later on. We will show that **tangents at such points, if there is a tangent, cannot have a positive or a negative slope (and hence, it**

**necessarily must have zero slope).** Before, however, we do so, we make the following observations. *We shall confine our attention to non-constant functions.**

Suppose a **function f has the maximum value at some point** $x_0$. Then common-sense tells us **that the graph of the function must be below the line** $y=f(x_0)$. (See Fig. 26.6)

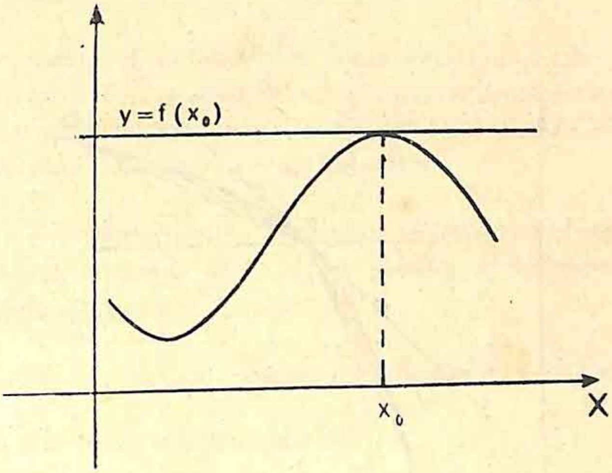

Fig. 26.6

**Similarly if f has the minimum value at some point** $x_0$, **then the graph of the function must**

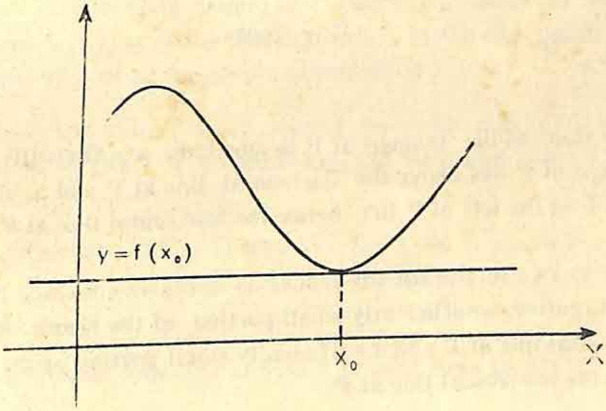

Fig. 26.7

**lie above the line** $y=f(x_0)$. (See Fig. 26.7) We now return to our problem:

---

*A constant function *f* is deemed to have the maximum value at every point of its domain and similarly the minimum value at every point of its domain.

Let us suppose the tangent has a positive slope. How does graph of $f$ behave when tangent at a point, say $P$, is positive? It is clear that secants $PQ$ for positions of $Q$ on the right of $P$ and sufficiently close to $P$ along the curve must point upwards and the secants $PQ_1$ for positions of $Q_1$ on the left of $P$ and sufficiently close to $P$ along the curve must point downwards. (See Fig. 26.8 )

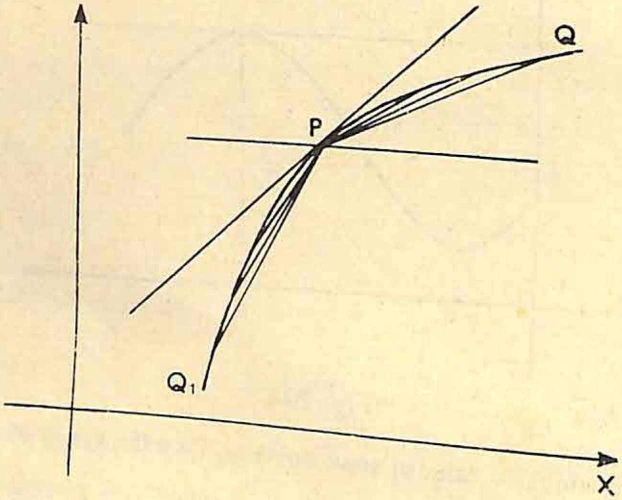

Fig. 26.8

**Thus, if the slope of the tangent at P is positive, a sufficiently small portion of the graph of f on the right of P lies above the horizontal line at P and a sufficiently small portion of the graph of f on the left of P lies below the horizontal line at P.**

We leave it as an exercise for the reader to similarly conclude that **if the slope of the tangent at P is negative, a sufficiently small portion of the graph of f on the right of P lies below the horizontal line at P and a sufficiently small portion of the graph of f on the left of P lies above the horizontal line at P.**

But at a point $x_0$ of maximum, the graph of $f$ is totally below the horizontal line $y = f(x_0)$. And, at a point of minimum, the graph of $f$ is totally above the horizontal line $y = f(x_0)$. Thus the slope of the tangent at the point of maximum or minimum can neither be positive nor negative—it must be zero.

We state this important result in the form of a theorem :

**Theorem 25 : Let f be a function defined on an interval J and let f attain its maximum or minimum value at some point $x_0$ in J ($x_0$ is not an end point of J). If $f'(x_0)$ exists, then**

$$f'(x_0)=0$$

Thus to look for points of maximum or minimum in the interior of J, we solve $f'(x) = 0$. The solutions of the equation $f'(x) = 0$ give us candidates for the maximum or minimum. But what about the end points ? We calculate the function-values at the end points and thus have the following **Working Rule, WR 3.**

**WR 3 : Let f be a continuous function* defined on an interval J of the type $a \leqslant x \leqslant b$ and let f be differentiable at all points of J, except possibly at end points. Then to find the maximum or minimum value of f,**

1. **Find all solutions of $f'(x)=0$. Denote the solutions by $x_1$, $x_2$, etc.**
2. **Compute the function-values at $x_1$, $x_2$, etc.**
3. **Compute the function-values at the end points.**

**Then the maximum value of f is the largest of the function values at the end points and at $x_1$, $x_2$, etc. The corresponding points are called the points of maximum. And the minimum value of f is the smallest of the function-values at the end points and at $x_1, x_2$, etc. The corresponding points are called the points of minimum.**

That a continuous function f does indeed have the maximum or minimum value in an interval J of the type $a \leqslant x \leqslant b$ is ensured by the **Extreme-Value Theorem,** which we state without proof.

**Theorem 26 : (Extreme-Value Theorem) : Let f be a continuous function on an interval J of the type $a \leqslant x \leqslant b$. Then f has the maximum value and attains it at least once in J.**

**Also f has the minimum value and attains it at least once in J.**

We consider some examples. We start with our favourite **Number Game** and further modify the restriction on numbers.

---

*Most of the non-constant functions that we deal with in this book are not constant even on any sub-interval.

**Example 1 :** *How should we choose two numbers, each greater than or equal to* $-2$, *whose sum is* $\frac{1}{2}$ *so that the sum of the square of the first and cube of the second is the minimum ?*

**Solution :** We again let the numbers be $x$ and $\frac{1}{2}-x$. Since both $x \geqslant -2$ and $\frac{1}{2}-x \geqslant -2$, **it is clear that x can take values in** $-2 \leqslant x \leqslant \frac{5}{2}$.

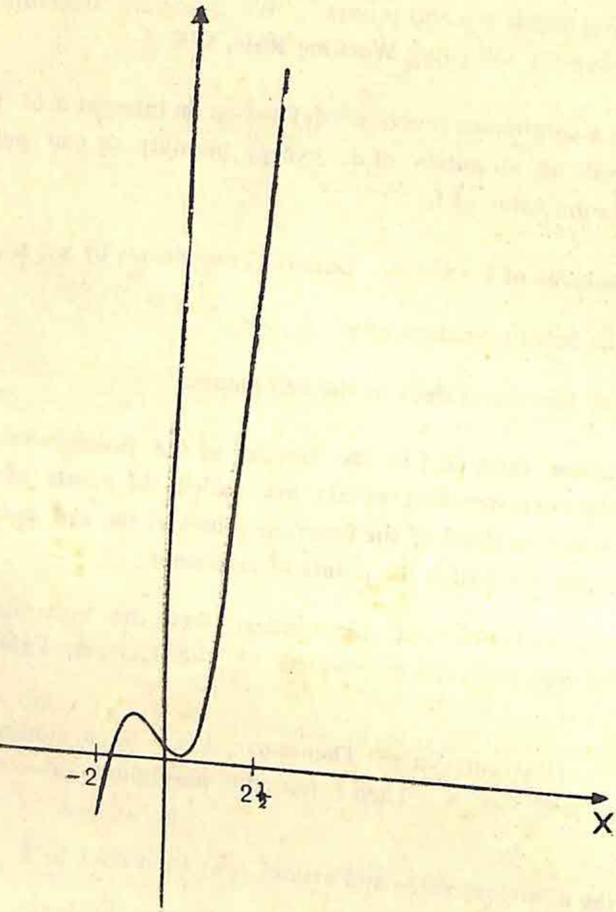

Fig. 26.9 : Graph of $y=\left(\frac{1}{2}-x\right)^2+x^3, -2 \leqslant x \leqslant \frac{5}{2}$

We have to find the minimum value of

$$f(x)=\left(\frac{1}{2}-x\right)^2+x^3=\frac{1}{4}-x+x^2+x^3$$

We find $f'(x)$ and then solutions of $f'(x)=0$. We have

$$f'(x)=-1+2x+3x^2=3\left(x-\frac{1}{3}\right)(x+1)$$

Thus $f'(x)=0$ when $x=\frac{1}{3}$ or $x=-1$. **$x=\frac{1}{3}$ and $x=-1$ are, therefore, candidates for the point of minimum.**

We compute the values of $f$ at the end-points and at $x=\frac{1}{3}$ and $-1$ and have the following table:

| $x$ | $-2$ | $-1$ | $\frac{1}{3}$ | $\frac{5}{2}$ |
|---|---|---|---|---|
| $f(x)$ | $-\frac{7}{4}$ | $\frac{5}{4}$ | $\frac{7}{108}$ | $\frac{157}{8}$ |

By *WR* 3, therefore, $f$ has the minimum value at $x=-2$. (See Fig. 26.9)

The required numbers are, therefore, $\frac{5}{2}$ and $-2$.

**Example 2:** Determine* the maximum and minimum of $f(x)=\cos x+\sin x,\ 0\leqslant x\leqslant\pi$

**Solution:** We have

$$f(x)=\cos x+\sin x$$

Thus, $f'(x)=-\sin x+\cos x$

$f'(x)=0$ when $\cos x=\sin x$, i.e., when $x=\frac{\pi}{4}$

---

*To find the maximum (minimum) of $f(x)$ means to find the maximum value (minimum value) of $f(x)$.

We have

$$f(0)=1$$

$$f\left(\frac{\pi}{4}\right)=\sqrt{2}$$

$$f(\pi)=-1$$

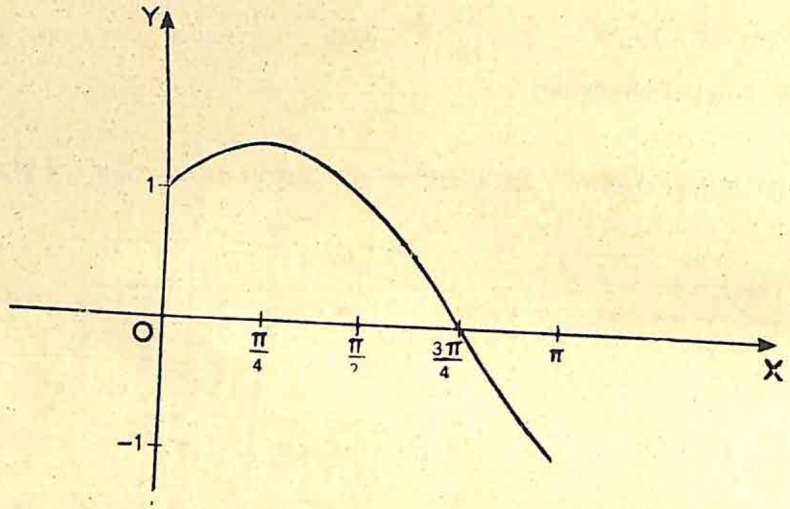

Fig. 26.10 : Graph of $y=\sin x+\cos x$, $0\leqslant x\leqslant \pi$

Thus, $f$ has the maximum at $x=\frac{\pi}{4}$ and the minimum at $x=\pi$. The maximum of $f$ is $\sqrt{2}$ and the minimum of $f$ is $-1$.

**Example 3**: Find the maximum and minimum of

$$f(x)=x^3-12x^2+36x+17 \text{ in } 1\leqslant x\leqslant 10$$

**Solution :** We have

$$f(x)=x^3-12x^2+36x+17$$

Thus, $f'(x)=3x^2-24x+36=3(x^2-8x+12)$

i.e., $f'(x)=3(x-6)(x-2)$

Thus, $f'(x)=0$ when $x=6$ or $x=2$

We have

$$f(1)=42,\ f(2)=49,\ f(6)=17,\ f(10)=177$$

Thus, $f$ has the maximum at $x=10$ and the minimum at $x=6$. The maximum of $f$ is 177 and the minimum of $f$ is 17.

[The reader is advised to draw a graph of the given function.]

**Example 4**: *The* **Demand Curve** *expresses a relationship between the number of units demanded of a particular item and the corresponding price.*

The **Demand Curve** of an item is given by the equation*

$$f(x)=31-0.25x^2,\ 0\leqslant x\leqslant 8$$

The **Cost Function** (in rupees) for $x$ items ($x\geqslant 0$) is given by

$$C(x)=4(x+1)$$

Determine the value of $x$ for which the firm makes the maximum profit.

**Solution**: It is clear that the Revenue $R(x)$ is the amount received by selling $x$ items. Thus,

$$R(x)=x\,f(x)=x(31-0.25\,x^2)$$

i.e., $R(x)=31x-0.25x^3,\ 0\leqslant x\leqslant 8$

We are given that the cost of producing these $x$ items is

$$C(x)=4(x+1)$$

Thus, the Profit Function $P(x)$ is

$$P(x)=R(x)-C(x)$$

$$=31x-0.25x^3-4(x+1)$$

i.e., $P(x)=27x-0.25x^3-4,\ 0\leqslant x\leqslant 8$

To find the maximum value of $P(x)$, we compute $P'(x)$. We have

$$P'(x)=27-0.75x^2$$

Now $P'(x)=0$ when $x^2=36$ i.e., when $x=\pm 6$ (clearly $x\neq -6$).

---

*Being an economic variable, 'number of units', $x$ takes only positive integral values. Further, $x=0$ is not economically plausible. However, viewing as a mathematical problem, we let $x$ take all values in the interval $0\leqslant x\leqslant 8$.

The candidates for the maximum are $x=6$, and, of course, the end-points. We have

$$P(0) = 4$$

$$P(6) = 104$$

$$P(8) = 84$$

Thus, the maximum profit is obtained by selling 6 items. The maximum profit is Rs 104.

Example 5 **Show that, of all the rectangles with a given perimeter, the square has the largest area**

**Solution:** Let us suppose that the given perimeter is $P$ metres, and let $x$ metres and $y$ metres be the dimensions of the rectangle.

Then, $2x+2y=P$

Thus, $$y=\frac{1}{2}(P-2x) \qquad (1)$$

Since $x\geqslant 0$, $y\geqslant 0$, it is clear that $0\leqslant x\leqslant\frac{P}{2}$

Now, Area$=xy = \frac{1}{2}x(P-2x)$

Let, $$A(x)=\frac{1}{2}Px-x^2,\quad 0\leqslant x\leqslant\frac{P}{2}$$

We find solutions of $A'(x)=0$. We have

$$A'(x)=\frac{1}{2}P-2x$$

Now, $A'(x)=0$ when $P=4x$ i.e., when $x=\frac{P}{4}$

Also, $A(0)=0$. $A\left(\frac{P}{4}\right)=\frac{P^2}{8}-\frac{P^2}{16}=\frac{P^2}{16}$

and. $A\left(\frac{P}{2}\right)=\frac{P^2}{4}-\frac{P^2}{4}=0.$

Thus, the area function $A(x)$ attains the maximum value at $x=\frac{P}{4}$

But when $P=4x$, we have, from (1), that $y=\frac{1}{2}(2x)=x$ which means that the dimensions of the rectangle (when it has maximum area) are $\frac{P}{4}$ metres and $\frac{P}{4}$ metres.

This proves our result, namely, that of all the rectangles with a given perimeter the square has the largest area.

[**Remark** : If instead of rectilinear figures, we allow closed curves with a given perimeter (length), it can be proved that the circle has the largest area. However the proof is beyond the scope of this book.]

## EXERCISE 26.2

Determine the maximum value and minimum value of each of the following in the stated domains :

**1.** $y=4x-\frac{1}{2}x^2, \quad -2\leqslant x\leqslant \frac{9}{2}$

**2.** $y=\frac{1}{2}x^2+5x+\frac{3}{2}, \quad -6\leqslant x\leqslant -2$

3. $f(x)=2x^2-9x-1, \quad 2\leqslant x\leqslant 5$

4. $f(x)=x^3-\frac{5}{2}x^2-2x+1, \quad 0\leqslant x\leqslant 3$

5. $y=\frac{2}{3}x^3+\frac{x^2}{2}-x+4, \quad 0\leqslant x\leqslant \frac{3}{2}$

6. $f(x)=x^3-48x+17, \quad 0\leqslant x\leqslant 5$

7 $y=x^4-8x^2+16, \quad -4\leqslant x\leqslant 0$

**8.** $f(x)=(x+1)^{\frac{2}{3}}, \quad 0\leqslant x\leqslant 8$

[**Hint** : If there are no solutions to $f'(x)=0$, then the only candidates for the maximum and the minimum are the end-points.]

9. $f(x)=\dfrac{x+1}{\sqrt{x^2+1}}$, $\quad 0\leqslant x\leqslant 2$

10. $y=x+\sin 2x$, $\quad 0\leqslant x\leqslant 2\pi$

11. $y=-x+2\sin x$, $\quad 0\leqslant x\leqslant 2\pi$

12. $f(x)=[x(x-1)+1]^{\frac{1}{3}}$, $\quad 0\leqslant x\leqslant 1$

13. $g(x)=x^2\sqrt{1+x}$, $\quad -1\leqslant x\leqslant \dfrac{1}{2}$

14. $y=2\cos 2x-\cos 4x$, $\quad 0\leqslant x\leqslant \pi$

16. Show that the height of the right circular cylinder of **maximum volume** that can be inscribed in a given right circular cone of height $h$ is $\frac{1}{3}h$.

17. Find the maximum value of

$$y=x^{\frac{2}{3}}+(x-2)^{\frac{2}{3}}$$

in $0\leqslant x\leqslant 2$.

18. Find the area of the largest rectangle having the perimeter of 200 metres.

### 26.5 The Extended Number Game : Local Maxima and Minima

We return to our example of the **Number Game** and relax the restrictions on numbers. *How should we choose two numbers whose sum is $\frac{1}{2}$ so that the sum of the square of the first and the cube of the second is the minimum* ?

As before, we let $\frac{1}{2}-x$ and $x$ be the two numbers. Then we have to minimize

$$f(x)=\left(\frac{1}{2}-x\right)^2+x^3=\frac{1}{4}-x+x^2+x^3$$

We have $f'(x)=-1+2x+3x^2=3\left(x-\frac{1}{3}\right)(x+1)$

Where does $f$ have the minimum now ?

We study the sign of $f'$ and have the following situation :

| | | |
|---|---|---|
| $x < -1$ | $f'$ positive | $f$ increasing |
| $-1 < x < \frac{1}{3}$ | $f'$ negative | $f$ decreasing |
| $\frac{1}{3} < x$ | $f'$ positive | $f$ increasing |

A pertinent question that arises now is **how far does f decrease?** It can be proved* that $f(x) \to -\infty$ as $x \to -\infty$ and that $f(x) \to \infty$ as $x \to \infty$. A graph of $f$ is given in Fig. 26.11.

We see, therefore, that **f has no point of minimum.** [In fact, it is easy to see that **f also has no point of maximum.**] What exactly is happening ?

We have the following situation :

| $y = \left(\frac{1}{2} - x\right)^2 + x^3$ | |
|---|---|
| $0 \leqslant x \leqslant \frac{1}{2}$ | Point of minimum at $x = \frac{1}{2}$ |
| $-\frac{3}{2} \leqslant x \leqslant 2$ | Point of minimum at $x = \frac{1}{3}$ |
| $-2 \leqslant x \leqslant \frac{5}{2}$ | Point of minimum at $x = -2$ |
| $x$ real | No point of minimum |

---

* In fact, we can prove a stronger result :

(i) If f (x) is an **odd-degree** polynomial in x,
$f(x) \to \infty$ as $x \to \infty$ and $f(x) \to -\infty$ as $x \to -\infty$.

(ii) If f (x) is an **even-degree** polynomial in x,
$f(x) \to \infty$ as $x \to \infty$ or $x \to -\infty$

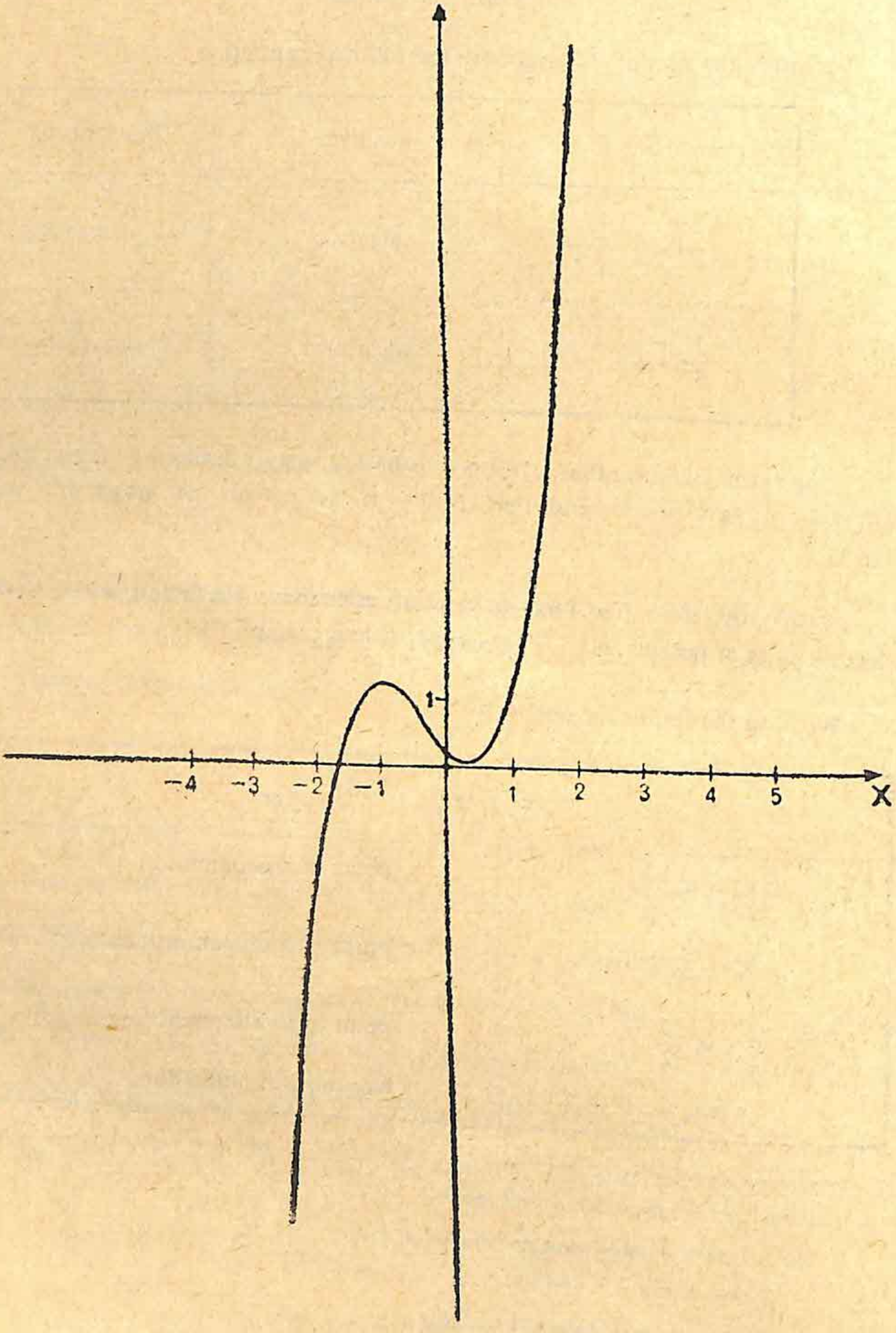

Fig. 26 14: Graph of $y = \left(\frac{1}{2} - x\right)^2 + x^3$

We see, therefore, that $x=\frac{1}{3}$ is the point of minimum in $-\frac{3}{2} \leqslant x \leqslant 2$ but that in the enlarged domain $-2 \leqslant x \leqslant \frac{5}{2}$, it is no longer the point of minimum; now $x=-2$ becomes the point of minimum. Further that if we let $x$ be real, then $f$ has no point of minimum at all. We say, $\mathbf{x = \frac{1}{3}}$ **is a point of local minimum** in the sense that it is a point of minimum for $f$ in some interval containing $x=\frac{1}{3}$ of which $\frac{1}{3}$ is not an end-point.

We now make some definitions :

**Definition 6 : A function defined on an interval J has a local maximum (value) at a point $x_0$ in J if for some interval around* $x_0$, $f(x_0) > f(x)$ for all x in this interval. $x_0$ is said to be a point of local maximum.**

**Definition 7 : A function defined on an interval J has a local minimum (value) at a point $x_0$ in J if for some interval around $x_0$, $f(x_0) < f(x)$ for all x in this interval. $x_0$ is said to be a point of local maximum**

[ The *plurals* of *maximum* and *minimum* are *maxima* and *minima respectively.* ]

**Remark :** In the extended number game, clearly **x= −2** is **not a point of local minimum,** even though it is the point of minimum in the interval $-2 \leqslant x \leqslant \frac{5}{2}$

**Now how to determine these local maxima and minima ?** Again, we consider only those functions which are continuous in J and differentiable at all points in J, except, possibly at end points. It is clear that if $x_0$ is a point of local maximum ( or minimum ), then $x_0$ is the point of maximum ( or minimum ) in some interval around $x_0$. Thus, by Theorem 25, $f'(x_0)=0$.

*But $f'(x_0)$ may be equal to zero and yet $f$ may not have local maximum or minimum at $x_0$.* Can you think of an example ?

[Consider $f(x)=x^3$ and $x_0=0$. Then $f'(0)=0$. But we recall that $y=x^3$ is an increasing function and, therefore, can not have any point of local maximum or minimum.]

**We see thus that $f'(x_0)=0$ is a necessary but not sufficient condition for our functions to have local maximum ( or minimum ) at $x_0$. So how should we be sure ?**

---

*This means that $x_0$ is not an end-point of this interval.

One thing is sure that if for an interval $J_0$ around $x_0$, **$f'(x) > 0$ for $x < x_0$ in $J_0$** and $f'(x) < 0$ for $x > x_0$ in $J_0$, then by **WRI** $x_0$ **is the point of maximum of $f$ in $J_0$** and hence, a point of local maximum of $f$.

Since $f'(x_0) = 0$, it means that a sufficiently small portion of the graph of $f'$ on the right of the point ( $x_0$, 0 ) lies below the horizontal line at this point and a sufficiently small portion of the graph of $f'$ on the left of the point ( $x_0$, 0 ) lies **above the horizontal** line at this point. Recall that this would be possible if the tangent to the curve $y=f'(x)$ at the point ( $x_0$, 0 ) has a negative slope. This would be so **if the derivative of $f'(x)$ at $x_0$, which we call the second derivative of f at $x_0$ and denote* by $f''(x_0)$, is negative.**

We state this in the form of a **Working Rule, WR4 :**

**WR4 : If f is defined and continuous in some J and $x_0$ is in J, but not an end-point of J, and if**

(i) $f'(x_0) = 0$

(ii) $f''(x_0) < 0$,

**then, f has a local maximum at $x_0$.**

Similar reasoning for a point of local minimum gives us the following **Working Rule, WR5 :**

**WR5 : If f is defined and continuous in some J and $x_0$ is in J, but not an end-point of J, and if**

(i) $f'(x_0) = 0$

(ii) $f''(x_0) > 0$,

**then f has a local minimum at $x_0$.**

Local maxima and local minima are important in their own right. For instance, when we look around ourselves and see a tall building, we tend to believe that it is the tallest building around. However, when we extend our field of vision, we may find a still taller building.

We consider some examples.

**Example 1 :** Find the points of local maxima and minima of the function

$$f(x) = x\sqrt{32-x^2}$$

in $-5 \leqslant x \leqslant 5$.

---

*If we write our function as $y=f(x)$, another notation for the second derivative is $\frac{d^2y}{dx^2}$. The second derivative of $f$ can be computed at a point $x_0$ only if the first derivative $f'$ of $f$ can be computed in an interval around $x_0$.

**Solution :** We have

$$f(x) = x\sqrt{32-x^2}$$

Thus, $$f'(x) = x \cdot \frac{1}{2\sqrt{32-x^2}}(-2x) + \sqrt{32-x^2}$$

$$= \frac{-2x^2+32}{\sqrt{32-x^2}}$$

Now $f'(x) = 0$ when $-2x^2+32 = 0$, i. e., when $x = \pm 4$.

Thus there are two candidates for local maxima and minima. We must find $\frac{d^2y}{dx^2}$ at these points to determine which is which, if at all. We have

$$f''(x) = \frac{\sqrt{32-x^2}\,(-4x) - (-2x^2+32)\,\dfrac{(-2x)}{2\sqrt{32-x^2}}}{(\sqrt{32-x^2})^2}$$

$$= \frac{-4x\sqrt{32-x^2} - \dfrac{2x\,(x^2-16)}{\sqrt{32-x^2}}}{32-x^2}$$

Clearly, $f''(x)\Big]_{x=4} = --4$ and $f''(x)\Big]_{x=-4} = 4$

Thus, $f(x)$ has a local maximum at $x=4$ and a local minimum at $x= =4$.

**Example 2 :** Find the local maxima and local minima of

$$y = \sin x$$

for all real $x$.

**Solution :** We have

$$y = \sin x$$

Thus, $$\frac{dy}{dx} = \cos x$$

Now, $\cos x = 0$ when $x = \frac{\pi}{2} + n\pi$, where $n$ is an integer.

Now, $$\frac{d^2y}{dx^2} = -\sin x$$

We see that $\left.\dfrac{d^2y}{dx^2}\right]_{x=\frac{\pi}{2}} = -1$, $\left.\dfrac{d^2y}{dx^2}\right]_{x=\frac{5\pi}{2}} = -1$,

$\left.\dfrac{d^2y}{dx^2}\right]_{x=\frac{9\pi}{2}} = -1$, $\left.\dfrac{d^2y}{dx^2}\right]_{x=\frac{-3\pi}{2}} = -1$, $\left.\dfrac{d^2y}{dx^2}\right]_{x=\frac{-7\pi}{2}} = -1$, etc.

i.e., $\dfrac{d^2y}{dx^2} < 0$ at $x = \dfrac{\pi}{2}, \dfrac{5\pi}{2}, \dfrac{7\pi}{2}$, etc. and at $x = \dfrac{-3\pi}{2}, \dfrac{-7\pi}{2}$, etc.

i.e., $\dfrac{d^2y}{dx^2} < 0$ at $x = n\pi + \dfrac{\pi}{2}$ where $n$ is an even integer.

Hence $y = \sin x$ has local maxima at $x = n\pi + \dfrac{\pi}{2}$ where $n$ is an even integer. Each of the local maximum values is 1. (Why ?)

Similarly, we can show that $y = \sin x$ has local minima at $x = n\pi + \dfrac{\pi}{2}$ where *n is* an odd integer. Each of the local minimum values is $-1$. (Why ?)

**Example 3** : Find the local minima of $f(x) = x^4$ for all re

**Solution** : We have

$$f(x) = x^4$$

Thus, $$f'(x) = 4x^3$$

Now, $f'(x) = 0$ when $x = 0$. We find

$$f''(x) = 12x^2$$

At $x = 0$, $f''(x) = 0$.

We see that $f''(x)$ is neither positive nor negative at $x = 0$. **WR5**, therefore, fails to tell us anything about local minima. However, $f'(x) > 0$ for $x > 0$ and $< 0$ for $x < 0$. Thus, by **WR1**, $x = 0$ is the point of minimum and hence, a point of local minimum as well. (**Why** ?)

**Remark 1** : **WR5** and **WR4** are sufficient but not necessary in the sense that there may be points of local maxima and local minima which can not be found by the use of these rules.

**Remark 2** : It should be clear that **WR4** and **WR5** can also be used to find the points of maximum and minimum, respectively. **If a function is defined and continuous on a closed interval J of the type $a \leqslant x \leqslant b$, then we find the local maxima and minima by WR4 and WR5 and check the function-values at the end-points as well.**

We consider some examples.

**Example 4** : A cannon is fired at an angle $\theta \left(0 \leqslant \theta \leqslant \frac{\pi}{2}\right)$ with the horizontal. If $v$ is the initial velocity of the cannon ball, the height $h$ of the ball at time $t$, ignoring wind resistance, is given by

$$h = (v \sin \theta)t - 4.9t^2$$

(a) When will the ball return to the ground ?

(b) How far will the ball have travelled horizontally at the time it hits the ground, assuming there are no forces in the horizontal direction ?

(c) Determine $\theta$ so that the horizontal range of the ball is maximum.

**Solution** : (a) When the ball returns to the ground, clearly $h = 0$

i.e., $vt \sin \theta - 4.9\, t^2 = 0$

or, $t(v \sin \theta - 4.9t) = 0$

Thus, $t = 0$ or $\dfrac{v \sin \theta}{4.9}$

Clearly $t \neq 0$. Thus the ball will return to the ground after $\dfrac{v \sin \theta}{4.9}$ units of time it was fired.

(**b**) Since there is no wind resistance and there are no forces in the horizontal direction, it is intuitively clear that the horizontal component of the velocity is $v \cos \theta$ all through. Thus, the horizontal distance travelled by the ball in $t$ units of time is

$$(v \cos \theta)t$$

when it hits the ground, then the horizontal distance travelled is

$$(v \cos \theta)t \Big]_{t = \frac{v \sin \theta}{4.9}} = \frac{v^2 \sin \theta \cos \theta}{4.9}$$

Let us denote this distance by R.

(i) We wish to find $\theta$ so that $R$, as a function of $\theta$, is maximum. We have

$$R = \frac{v^2 \sin \theta \cos \theta}{4.9} = \frac{v^2 \sin 2\theta}{9.8}$$

Now, $$\frac{dR}{d\theta} = \frac{v^2 \cos 2\,\theta}{4.9}$$

$\frac{dR}{d\theta} = 0$ when $\cos 2\theta = 0$. But $\theta$ is in $0 \leqslant \theta \leqslant \frac{\pi}{2}$, thus $\cos 2\theta = 0$ when $\theta = \frac{\pi}{4}$

Next we find $\frac{d^2R}{d\theta^2}$ and see if it is maximum for $\theta = \frac{\pi}{4}$. We have

$$\frac{d^2R}{d\theta^2} = -\frac{2v^2 \sin 2\theta}{4.9}$$

$$\frac{d^2R}{d\theta^2}\Big]_{\theta = \frac{\pi}{4}} = \frac{-2v^2 \sin 2\theta}{4.9}\Big]_{\theta = \frac{\pi}{4}} < 0$$

Thus, $R$ is maximum when $\theta = \frac{\pi}{4}$. In other words, If the cannon is fired at an angle $\frac{\pi}{4}$ to the ground, it will have the maximum range.

**Example 5 :** Find the maxima and minima of

$$f(x) = x\sqrt{32 - x^2}$$

in $-5 \leqslant x \leqslant 5$. [ See also Example 1, Section 26.5]

**Solution :** We have seen in Example 1, that there is a local maximum at $x = 4$ and a local minimum at $x = -4$.

We calculate the local maximum and local minimum values. We have

$$\text{Local maximum value} = f(x)\Big]_{x=4} = 16$$

$$\text{Local minimum value} = f(x)\Big]_{x=-4} = -16$$

We then check the function-values at the end-points. We have

$$f(-5) = -5\sqrt{7}$$

And, $$f(5) = 5\sqrt{7}$$

We see that $f(5) < f(4)$. Thus $x=4$ is the point of maximum and the maximum value is 16. Similarly we see that $x=-4$ is the point of minimum and the minimum value is $-16$.

## EXERCISE 26.3

Find the points of local maxima and local minima, if any, of each of the following functions. Find also the local maximum values and local minimum values.

1. $f(x) = x^3-6x^2+9x+15,\ 0 \leqslant x \leqslant 6$
2. $f(x) = (x-1)(x+2)^2$
3. $y=\dfrac{x}{2}+\dfrac{2}{x},\ x>0$
4. $g(x)=\dfrac{1}{x^2+2}$
5. $y=\sqrt[3]{x^2}-4,\ x>0$
6. $y=x\sqrt{1-x},\ x\leqslant 1$
7. $f(x)=\sin^4 x+\cos^4 x,\ 0< x < \dfrac{\pi}{2}$
8. $y=\sin 2x-x,\ -\dfrac{\pi}{2} \leqslant x \leqslant \dfrac{\pi}{2}$
9. $f(x) = \dfrac{2}{x}-\dfrac{2}{x^2},\ x > 0$
10. $y=\dfrac{x}{(x-1)(x-4)},\ 1< x <4$
11. $y=\dfrac{x^2-5}{x-3},\ 0\leqslant x \leqslant 2$

12. Find the local maximum and local minimum values of

$$y=(x+1)(x+2)^{\frac{1}{3}}$$

in $x \geqslant -2$.

**13.** Determine two positive numbers whose sum is 15 and the sum of whose squares is minimum.

***14**. A wire of length 28 m is to be cut into two pieces. One of the pieces is to be made into a square and the other into a circle. Where should the wire be cut so that combined area is minimum?

[**Hint**: Let $x$ metres be made into a circle and $(28-x)$ metres into a square. Then the radius of the circle is $\frac{x}{2\pi}$ and a side of the square is $\frac{28-x}{4}$.]

## 26.6 Key Concepts

The point of maximum

The point of minimum

The maximum value of a function

The minimum value of a function

Extreme Value Theorem

Points of local maxima and minima

Second derivative of a function

## 26.7 Suggestions for Further Reading

*Any standard text book on Calculus will have a chapter on 'maxima and minima'.* The reader is referred, for instance, to [1] Section 18.7, [1] Section 21.9.

*A rigorous treatment of maximum and minimum problems, at a rather advanced level, is given in*

[1] J. L. Walsh: **A Rigorous Treatment of Maximum-Minimum Problems in the Calculus**

D.C. Heath and Company, Boston, (U.S.A.). 1962

## MISCELLANEOUS EXERCISE VI

### (On Units XXIV, XXV and XXVI)

1. Show that $f(x)=x-\frac{1}{2}$ is increasing for all real values of $x$.

2. Prove that $u(x)=mx+b$, where $m$ and $b$ are constants, and $m<0$, is a decreasing function of $x$ for all real $x$.

3. Show that $y=-\frac{1}{x}+1$ is increasing for all real $x>0$. Show also that it is increasing for all real $x<0$.

4. Show that $y=-\frac{1}{x}+1$ is neither increasing nor decreasing in $-\infty<x<\infty$

5. Determine the intervals where

$$f(x)=\sin x-\cos x,\ 0\leqslant x\leqslant 2\pi$$

is increasing or decreasing.

*Show that the following functions are increasing in the indicated intervals :*

6. $f(x)=\frac{x}{1+x}$, $x>1$

7. $f(x)=x+\frac{1}{x}$, $x\geqslant 1$

*8. $y=x^9+4x^7+11$, for all real $x$.

*9. $f(x)=x^2-x\sin x$, $0\leqslant x\leqslant\frac{\pi}{2}$

[**Hint :** Show that $f'(0)=0$ and $f'$ is increasing.]

*Show that the following functions are decreasing in the indicated intervals :*

10. $f(x)=-7x^2+11x-9$, $x>1$

**11.** $f(x) = x^{-5} + \frac{1}{2} x^{-3} + 1, x > 0$

**12.** $y = \frac{1}{x}, x < 0$

13. $f(x) = x^3 + \frac{1}{x^3}, 0 < x < 1$

*Determine for which values of x, the following functions are increasing and for which values they are decreasing :*

**14.** $y = x^4 - \frac{x^3}{3}$

**15.** $y = x^6 + 3x^4 + 2x^2 + 110$

**16.** $f(x) = 5x^{\frac{3}{2}} - 3x^{\frac{5}{2}}, x > 0$

**17.** $g(x) = \frac{x-2}{x+1}, \quad x \neq -1$

*Sketch a graph of each of the following functions*:

**18.** $f(x) = |x - 3|$

**19.** $g(x) = [x], \quad 0 \leqslant x \leqslant 3$

$[x]$ denotes the greatest integer $\leqslant x$ and is called the integral part of $x$.

**20.** $y = x + \frac{4}{x}, x \neq 0$

**21.** $f(x) = x^2(4x - 3) + 5, \quad 0 \leqslant x < 1$

**22.** Consider $f(x) = \frac{x^3 - 1}{x - 1}, x \neq 1$. How should we define $f(1)$ so that $f$ is continuous ?

***23.** Discuss the continuity of $f(x) = [x], \quad 0 \leqslant x \leqslant 3$.

**24.** Prove that $f(x) = \frac{x^2 - 1}{3}$ is continuous at $x = 0$.

25. Prove that $g(x) = \begin{cases} \dfrac{\sin x}{x} + \cos x, & x \neq 0 \\ 2 & x = 0 \end{cases}$

is continuous at $x=0$.

*Evaluate the following limits :*

26. $\lim\limits_{x\to 0} \dfrac{\cos(5+x) - \cos(5-x)}{x}$

27. $\lim\limits_{x \to 0} \dfrac{\sin(1+x^2)-\sin(1-x^2)}{x}$

28. $\lim\limits_{y \to 0} \dfrac{(1+y^2)\sin y^3}{y^2}$

29. $\lim\limits_{t \to 0} \dfrac{\tan t}{(1-t^2)\sin t}$

30. Is the function

$$f(x) = \begin{cases} \dfrac{\sin 2x}{x}, & x \neq 0 \\ 2, & x=0 \end{cases}$$

continuous at $x=0$? Give reasons for your answer.

*Use the delta method to find the derivative of each of the following :*

31. $\cos x^2$

32. $\sin 3x$

*Find the derivative of each of the following :*

33. $\sin x^2 - \dfrac{\tan x}{1+x^2}$

34. $5\,\mathrm{cosec}^2 x + 3\cot x^2 + 5x$

35. $(\tan\sqrt{x} + x^2 - \sin x)^3$

36. $\sqrt{\dfrac{\sec x - 1}{\sec x + 1}}$

*Find the value of the derivative of each of the following at indicated point(s) :*

**37.** $\sin 2x+\cos 3x, \quad x=0, \quad \frac{\pi}{6}$

**38.** $\sin x^2+(1-x^2)\cos x^2, \quad x=\sqrt{\frac{\pi}{2}}$

39. $(\cot\sqrt{x}+5\sin^2\sqrt{x}+5x)^2, \; x=\frac{\pi^2}{16}$

*Find $\frac{dy}{dx}$ for each of the following :*

40. $y\cot x+y^3\tan x+\sin x=0$

**41.** $y\sin y-\cot x+xy^2=5$

**42.** $(1-y^2)\sin y+\tan y=5x$

*Find the equation of the tangent to each of the following at the indicated point :*

**43.** $y=\tan x+\sec x, \; x=\frac{\pi}{3}$

***44.** $y^2\sin x=9, \quad x=\frac{\pi}{2}$

**45.** $y\cot x=y^3\tan x, \quad x=\frac{\pi}{4}$

*Find the equation of the normal to each of the following at the indicated point :*

**46.** $y=\sin 2x+5\cos x. \quad x=\frac{\pi}{6}$

47. $x^2\tan y=\cot y, \; y=\frac{\pi}{4}$

**48.** $x^2\sin y-y^2\cos x+4=0, \quad x=0$

*Find the points of maximum and minimum of each of the following :*

**49.** $y=x(1+10x-x^2), \quad 3\leqslant x\leqslant 9$

51. $y=(x-1)^{\frac{1}{3}}(x-2), \quad 1 \leqslant x \leqslant 9$

52. $y=\sqrt{5}\left(\sin x+\frac{1}{2}\cos 2x\right), \quad 0 \leqslant x \leqslant \frac{\pi}{2}$

*53. For a given curved surface of a right circular cone when the volume is maximum, prove that the semi-vertical angle is $\theta$ where $\sin\theta=\frac{1}{\sqrt{3}}$

*54. When does the tangent to

$$f(x)=x^3-6x^2+9x+4$$

have the maximum slope in $0 \leqslant x \leqslant 5$ ?

55. The combined resistance, R, of two resistors is given by

$$\frac{1}{R}=\frac{1}{R_1}+\frac{1}{R_2}$$

where $R_1$ and $R_2$ are the respective resistances of the two resistors with $R_1+R_2=a$ $(R_1, R_2>0)$, where 'a' is a constant. Show that the maximum resistance R is obtained by choosing resistors for which $R_1=R_2$.

56. Determine two positive numbers whose sum is 30 and whose product is the maximum.

*Find the local maximum and local minimum, if any, of each of the following :*

57. $f(x) = x^4-62x^2+120x+9$

58. $f(x) = 2x^3-24x+107$

59. $y=x\sqrt{x}+5x+3\sqrt{x}, \quad x>0$

60. $f(x)=3x^{\frac{5}{3}}-5x, \quad x>0$

61. $f(x) = -x^3+12x^2-5$

62. $f(x) = \dfrac{x^2+2x-20}{x-4}, \quad 5<x<9$

63. $f(x) = x(33-x)^{\frac{1}{2}}$

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64. $f(x) = x^3 + \frac{3}{x}$, $0 < x < 2$

65. $f(x) = \sin x - \cos x$, $-\frac{\pi}{3} < x < 0$

66. The rate of a chemical reaction $y$, is given by the formula

$$y = kx(a-x)$$

where $x$ is the amount of product, a, is the amount of material at the beginning of the reaction and $k > 0$. Determine the value of $x$ for which the rate of chemical reaction will be the maximum.

67. A projectile is fired upwards. Its height above the surface of the earth at time $t$ is given by

$$h(t) = at^2 + bt + c$$

where $a$, $b$, $c$ are non-zero constants and $a < 0$. Determine how high will the projectile travel.

68. A firm has found from experience that its profit as a function of, $x$, the amount produced, is given by

$$p(x) = \frac{-x^3}{3} + 729\,x - 2500,\ 0 \leqslant x \leqslant 35$$

Find the value of $x$ that maximises the profit.

69. The rate of working of an engine is given by

$$f(v) = 15v + \frac{6000}{v},\ 0 < v < 30$$

where $v$ is the speed of the engine. Determine the value of $v$ for which the rate of working is least.

70. Determine the points on the curve

$$y = \frac{1}{4}x^2$$

which are nearest to the point (0,5)

## UNIT XXVII

# THE MATHEMATICS OF CHANCE

## (An Introduction to Probability)

*Statistics is the science of making inferences from data which are subject to uncertainty. A mathematical measure of uncertainty is* **probability.** *In this unit, we study some basic concepts of the Theory of Probability.*

### 27.1 Introduction

In our every day life, we often hear or say phrases, such as, "Probably it will rain today" or "Most probably, I will stand first in the examination" or that "Odds of a particular team winning are 3 : 1." These phrases involve an element of uncertainty. **How can we measure this uncertainty ?** A measure of uncertainty is provided by the branch of mathematics, called the **Theory of Probability.** We will learn some rudiments of this theory so as to be able to make clear the meaning and implications of such phrases.

The theory of probability had its origin in the 16th century. Oddly enough, it originated in games of chance, for instance, throwing of dice or coins, drawing cards from a deck or balls from an urn, etc. The first book on the subject was written by the **Italian mathematician, Jerome Cardan (1501-1576 A. D.).** The title of the book was **Book on Games of Chance** (*Liber de Ludo Aleae*) and it was published after his death, in 1663. Cardan came of very poor parents. He did not gamble but played dice almost daily for twenty five years and chess for forty years to escape from his poverty and chronic illnesses. He was a man of versatile qualities, scholar in mathematics and physics with a **degree** in medicine and was a well-known physician all over Europe.

A systematic study of probability, however, did not begin until the 17th century, when notable contributions were made by **the French mathematicians, Blaise Pascal (1623-1662 A.D. )** and **Pierre de Fermat (1601-1665 A.D.)** and **the Swiss mathematician, James**

**Bernoulli (1654-1705 A.D.).** Bernoulli's book, *Ars Conjectandi*, also published after his death, in 1713, is a major contribution to the Theory of Probability and Combinatorics.

The work done by the mathematicians of the 18th century, in particular, that of the **French mathematicians, Pierre-Simon de Laplace (1749-1827 A.D.)** set the pace for a tradition that dominated the theory of probability throughout the 19th century. Laplace's towering work, *Theorie analytique des probabilities* **(Theory of Analytical Probability)**, published in 1812, develops what is now called the **classical theory of probability**. Laplace's writings in mathematics and sciences are prolific. His last words are said to be "*What we know is very slight, what we don't know is immense.*"

Among the contributors in the 20th century, the **Russian mathematician**, A. N. **Kolmogorov** is credited with axiomatizing the theory. His small book *Foundations of Probability*, published in 1933, introduces probability, as a set-function and is considered a classic.

## 27.2 Probability : A Measure of Uncertainty

We turn our attention to one of the first problems that was responsible for development of the theory of probability, namely, that of throwing a die*. We throw** a die once, **What are all possible outcomes ?**

Clearly a die can fall with any of its faces uppermost. The number on each of the faces is, therefore, a possible outcome. We write **the set S, of all possible outcomes, as**

$$S=\{1, 2, 3, 4, 5, 6\}$$

The set S is called the **sample space** of the experiment (of throwing a die). 1, 2, 3, etc. are called **sample points** or simply, **points**.

Since our die is homogeneous (unbiased or balanced or "fair"), it is reasonable to assume that **each outcome is as likely to occur as any other outcome** in a throw of the die. We say that **outcomes are equally likely.** *Since there are six equally likely outcomes in a single throw of a die, we say, probability of any one face, for instance, 2, coming up is* $\frac{1}{6}$.

*We write,* $P(2) = \frac{1}{6}$

---

*A die is a homogeneous cube with six faces marked with numbers (dots) from 1 to 6. Plural of the word 'die' is dice'.

We assume that the die cannot fall on its edge or roll away.

This suggests a following definition of **probability**. *Unless otherwise specified, we assume that the outcomes of a sample space are equally likely.*

**Definition 8 : Probability of an event A, written as P(A), is defined* as**

$$P(A) = \frac{\textbf{Number of cases favourable to A}}{\textbf{Number of possible outcomes}}$$

**What is the probability of the die coming up with an odd number?** Since there are 3 odd numbers on the die, namely, 1, 3 and 5, we have

$$P(\text{odd number}) = \frac{3}{6} = \frac{1}{2}$$

**What is the probability of the die coming up with a number less than 5 ?** 1, 2, 3, and 4 are the only favourable cases. Thus

$$P(\text{number} < 5) = \frac{4}{6} = \frac{2}{3}$$

**Remark :** This definition of probability is circular in the sense that it involves "*equally-likely*" outcomes which, in turn, involve the concept of probability. However, we will only consider an intuitive meaning of the term '*equally-likely*' in the sense described already.

We consider some examples.

**Example 1 :** A coin is tossed once. What are all possible outcomes ? What is the probability of the coin coming up "*heads*"?

**Solution :** The coin can come up either "*heads*" (H) or "*tails*" (T). Thus the set S of all possible outcomes is

$$S = \{H, T\}$$

$$P(H) = \frac{1}{2}$$

---

*Using set-notation, we could write

$$P(A) = \frac{n(A)}{n(S)}$$

where n(S) denotes the number of elements (sample points) in the set S (sample space), and likewise, n(A) denotes the number of sample points in the set A (i.e., favourable to the event A.) It is clear that the event A is a subset of S.

**Example 2 :** A die is tossed once. What is the probability of the die coming up with "8"? What is the probability of the die coming up with a number $<8$?

**Solution :** Recall that there are 6 possible outcomes in a single toss of a die. Since no face is marked with 8 dots (or has the number 8), it is clear that

$$P(\text{a number} = 8) = \frac{0}{6} = 0$$

**We say that the probability of an impossible event is zero.**

Since every face of a die is marked with less than 8 dots (or has a number less than 8), it is clear that

$$P\,(\text{a number} < 8) = \frac{6}{6} = 1$$

We say that **the probability of a sure (certain) event*** is one.

It is clear from Example 2 that **the probability of an event A is a non-negative fraction never greater than 1.** We write

$$0 \leqslant P(A) \leqslant 1$$

**Example 3 :** In a simultaneous toss of two coins, what are all possible outcomes? Find the probability of '*2 heads*'. What is the probability of '*exactly 1 head*'?

**Solution : Clearly the coins must be distinguishable in the sense that we can speak of the first coin and the second coin.** We have

$$HH, \quad HT, \quad TH, \quad TT$$

as possible outcomes. (See Example 2, Section 7.2, Book I)

Thus, $P(\text{2 heads}) = \frac{1}{4}$, since only the outcome HH is favourable to the event, '*2 heads*'.

Now the event '*exactly 1 head*' is identical to the event '*exactly 1 head and 1 tail*'. (Why?) We see that there are two outcomes HT and TH that are favourable to this event. Hence,

$$P(\text{exactly 1 head}) = \frac{2}{4} = \frac{1}{2}$$

**Example 4 :** What is the probability of drawing an '*ace*' from a well-shuffled deck of 52 cards ?

**Solution :** *Well-shuffled ensures equally-likely outcomes.* There are 4 aces in a deck. Thus,

$$P(\text{an ace}) = \frac{4}{52} = \frac{1}{13}$$

---

*Sometimes, we write $P(S) = 1$.

**Example 5 :** From an urn (bag) containing 6 red and 3 black balls, a ball is drawn '*at random*'. What is the probability of drawing a '*red ball*' ?

**Solution :** *Random drawing of balls ensures equally-likely outcomes.* There are 6 red balls out of a total of $6+3=9$ balls. Thus,

$$\text{P(a red ball)} = \frac{6}{9} = \frac{2}{3}$$

**Example 6 :** In a single throw of two dice, what is the probability of obtaining '*a total of 9*' ?

**Solution :** *Again the dice must be distinguishable in the sense that we can speak of the first die and the second die.* How many outcomes are there in a single throw of two dice ? There are 36 possible outcomes. We write them below :

| | | | | | |
|---|---|---|---|---|---|
| 11 | 21 | 31 | 41 | 51 | 61 |
| 12 | 22 | 32 | 42 | 52 | 62 |
| ~~13~~ | 23 | 33 | 43 | 53 | 63 |
| 14 | 24 | 34 | 44 | 54 | 64 |
| 15 | 25 | 35 | 45 | 55 | 65 |
| 16 | 26 | 36 | 46 | 56 | 66 |

An outcome such as '*35*' means '*3 on the first die and 5 on the second die*' Clearly, it is different from the outcome '*53*'.

Now how do we get a *total of* 9 ? We have $3+6=9$, $4+5=9$, $5+4=9$, $6+3=9$. In other words, the outcomes 36, 45, 54 and 63 are favourable to the said event. Thus,

$$\text{P (a total of 9)} = \frac{4}{36} = \frac{1}{9}$$

## EXERCISE 27.1

1. An experiment results in four possible outcomes $s_1$, $s_2$, $s_3$ and $s_4$ with probabilities $p_1$, $p_2$, $p_3$ and $p_4$ respectively. State which of the following probability assignments are not possible. Give reasons for your answers.

   (a) $p_1=0.2$, $p_2=0.3$, $p_3=0.4$, $p_4=0.1$

   (b) $p_1=0.25$, $p_2=0.35$, $p_3=0.10$, $p_4=0.05$

(c) $p_1=0.4$, $p_2=-0.2$, $p_3=0.6$, $p_4=0.2$

(d) $p_1=0.3$, $p_2=0.6$, $p_3=0.1$, $p_4=0.1$

A die is thrown once. Find

(a) P (even number)

(b) P (a number $\geqslant 3$)

(c) P (a number $\leqslant 4$)

(d) P (a number $<7$)

(e) P (a number more than 6)

3. In a simultaneous toss of two coins, find the probability of

(a) 2 tails

(b) exactly 1 tail

(c) no tails

4. A card is drawn at random from a well-shuffled deck of 52 cards. Find the probability of

(a) a king

(b) a heart

(c) a seven of heart

(d) a jack. queen or a king

(e) a two of heart or diamond

(f) a black card

5. Three coins are tossed once.

(a) Write the points of S.

(b) Find the probability of

(i) 3 heads

(ii) exactly 2 heads

*(iii) at least 2 heads

*(iv) at most 2 heads

(v) no heads

*6. Four coins are tossed once. There are 16 possible outcomes, for instance, HHHH, HHHT, HHTH, HHTT, etc. Find

(a) P (4 tails)

(b) P (exactly 3 tails)

(c) P (exactly 2 tails)

(d) P (at least 1 tail)

7. From a bag containing 7 red and 5 black marbles, one is drawn at random. What is the probability of the marble being black ?

8. An urn contains 9 red, 7 white and 4 black balls. A ball is drawn at random. What is the probability that the ball drawn will be

(a) red ?

(b) white ?

*(c) red or black ?

(d) white or black ?

*(e) not red ?

9. A bag contains 3 red balls each bearing one of the numbers 1, 2 and 3 ; and 2 black balls each bearing one of the numbers 4 and 6. A ball is drawn, its number is noted and the ball is replaced in the bag. Then another ball is drawn and its number is noted.

(a) Write the points of S.

(b) Find the probability of drawing

(i) 2 on the first draw and 6 on the second draw

(ii) a number $\leqslant 2$ on the first draw and 4 on the second draw

(iii) a total of 5

**10.** In a single throw of two dice, find

(a) P (odd number on the first die and 6 on the second)

(b) P (a number $>4$ on each die)

*(c) P (a total of 11)

*(d) P (a total of 9 or 11)

*(e) P (a total $>8$)

## 27.3 Two Fundamental Theorems of Probability

Writing down the elements of *S* and counting the number of favourable cases often proves tedious in practical problems. For instance, if we were to draw 2 cards, replacing one before the second is drawn, there would be $52\times52=2704$ sample points. Or in three tosses of a die, there would be $6\times6\times6=216$ sample points. To write them is a formidable task and very often unnecessary. We study **two fundamental theorems of probability** which will greatly facilitate computation of probabilities for such experiments. Before, however, we do so, we make the following definition.

**Definition 9 : Two events are said to be mutually exclusive if the occurrence of one precludes (rules out) the simultaneous occurrence of the other.** In other words, *mutually exclusive events cannot occur simultaneously.*

It is clear from the definition, therefore, that *if A and B are mutually exclusive events, then P (A and B) = 0.*

Can you think of some examples of mutually exclusive events ? The event of *'getting an odd number'* and the event of *'getting an even number'* on a throw of a die are mutually exclusive.

We state now an important theorem that tells us that *the probability of one or the other of two mutually exclusive events is the sum of their separate probabilities.*

**Theorem 27 : If A, B are two mutually exclusive events, then**

$$\mathbf{P\,(A \text{ or } B) = P\,(A) + P\,(B)}$$

The proof is rather simple. Clearly, the number of cases favourable to the event *'A or B'* is the total of the number of cases favourable to the event *'A'* and those favourable to the event *'B'*.

We consider some examples.

**Example 1 :** In a single threw of two dice, find the probability of *a total of 9 or 11.* [See also Q. 10(d), Exercise 27.1]

**Solution :** Clearly, the events *a total of 9* and *a total of* 11 are mutually exclusive. (Why ?)

$$\text{Now, P (a total of 9)} = \text{P (36, 45, 54, 63)} = \frac{4}{36} = \frac{1}{9}$$

$$\text{And, P (a total of 11)} = \text{P (56,65)} = \frac{2}{36} = \frac{1}{18}$$

Thus, by Theorem 27,

$$\text{P (a total of 9 or 11)} = \text{P (a total of 9)} + \text{P (a total of 11)}$$

$$= \frac{1}{9} + \frac{1}{18} = \frac{3}{18} = \frac{1}{6}$$

**Example 2 :** The probabilities* that a student will receive an A, B, C or D grade are 0.30, 0.35, 0.20 and 0.15 respectively. What is the probability that a student will receive *at least a B grade* ?

**Solution** : The event *'at least a B grade'* means that the student gets *'either a B grade or an A grade'*.

$$\text{Thus, } P(\text{at least a B grade}) = P(\text{B grade}) + P(\text{A grade})$$

$$= 0.35 + 0.30$$

$$= 0.65$$

**Example 3 : Prove that the probability of the non-occurrence of an event '*A*' is 1−P (A), i.e.,**

$$\mathbf{P\ (not\ A) = 1 - P\ (A)}$$

---

* To be able to apply Theorem 27, it is not necessary that the events be equally likely.

**Solution** : We recall from Section 27.2 that the probability of the sample space S, in any experiment, is 1.

Now it is clear that, if in an experiment an event $A$ occurs, then the event *not A* cannot occur simultaneously, i.e., the two events are mutually exclusive. Also, the sample points of the two events together constitute the sample space S.

Thus, $P(A \text{ or not } A) = P(S) = 1$

i.e., $P(A) + P(\text{not } A) = 1$

Thus, $P(\text{not } A) = 1 - P(A)$

which proves the result.

**Example 4** : Find the probability of the event *at least 1 tail* if four coins are tossed once. [See also Q. 6(*d*), Exercise 27.1]

**Solution** : $P(\text{at least 1 tail}) = P(1 \text{ or } 2 \text{ or } 3 \text{ or } 4 \text{ tails})$

$$= 1 - P\,(0 \text{ tails}) = 1 - P\,(HHHH)$$

$$= 1 - \frac{1}{16} = \frac{15}{16}$$

**Remark** : We could have computed P (1 tail), P (2 tails), P (3 tails) and P (4 tails) and found their sum to obtain $P$(at least 1 tail). However, as is clear, that would have been much tedious.

**What if the events are not mutually exclusive ?** For instance, how would we find the probability of getting *either a king or a heart* from a well-shuffled deck of 52 cards ? Clearly, the two events, *a king* and *a heart* are not mutually exclusive since the outcome *a king of heart* is favourable to both the events.

We state below a theorem to find the probability of two such events.

**Theorem 28** : **If A, B are any two events, then**

$$\mathbf{P\,(A \text{ or } B) = P\,(A) + P\,(B) - P\,(A \text{ and } B)}$$

The proof is left to the reader as an exercise.

**Remark** : We often refer to this theorem as the (General) **Addition Theorem of Probabilities.** It is clear that when $A$, $B$ are mutually exclusive events, Theorem 27 falls through as a special case of Theorem 28. Theorem 27 is often called the **(Special) Addition Theorem of Probabilities.**

We now return to our example of drawing *a king or a heart*. We have

$$\text{P(a king or a heart)} = \text{P(a king)}+\text{P(a heart)} - \text{P(a king and a heart)}$$

$$= \frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13}$$

We consider some more examples.

**Example 5:** Two dice are tossed once. Find the probability of getting *'an even number on the first die or a total of 8'.*

**Solution :** P(even number on first die or a total of 8)

$$= \text{P(even number on first die)} + \text{P(a total of 8)}$$

$$- \text{P(even number on first die and a total of 8)}$$

$$= \text{P}(21, 22, 23, 24, 25, 26, 41, 42, 43, 44, 45, 46, 61, 62, 63, 64, 65, 66)$$

$$+ \text{P}(26, 35, 44, 53, 62) - \text{P}(26, 44, 62)$$

$$= \frac{18}{36} + \frac{5}{36} - \frac{3}{36} = \frac{20}{36} = \frac{5}{9}$$

The **Addition Theorem** necessitates the calculation of probabilities of events, such as, '*A and B*'. It is not always practical to enumerate the outcomes favourable to such events and hence compute their probabilities. Can we have some easier method of computing these probabilities? Before we give a method, we make the following definition:

**Definition 10: Two events A, B are said to be independent* if the occurrence or non-occurrence of one does not affect the probability of the occurrence (and hence non-occurrence) of the other.**

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*Often students make a mistake in defining independent events. They drop the key word, probability of the occurrence of the other. Simply saying that the occurence of one event does not affect the occurrence of the other is not independence in statistical sense.

Can you think of some examples of independent events? The event of getting '*H*' on first coin and the event of getting '*T*' on the second coin in a simultaneous throw of two coins are independent. What about the event of getting '*H*' on the first throw and event of getting '*T*' on the second throw in two successive throws of a coin? They are also independent.

Can you think of some examples of events that are **not independent***? The event of '*drawing an ace*' and the event of '*drawing a king*' in two successive draws of a card, without replacement. from a well shuffled deck of 52 cards, are **not independent**. (Why?) Supposing we draw an ace in the first draw. Then the first event has occurred. Now we do not raplace the card. It affects the probability of drawing a king in the second draw— rather than $\frac{4}{52}$, the probability of a king in the second draw is $\frac{4}{51}$.

**Remark**: If the cards are drawn with replacement the two events become independent.

We state now, without proof, an important theorem to find the probability of simultaneous occurrence of two independent events.

**Theorem 29 : If A, B are independent events, then**

$$P(A \text{ and } B) = P(A)P(B)$$

*In words, the probability of simultaneous occurrence of two independent events is the product of their separate probabilities.*

We consider some examples.

**Example 6** : A die is tossed twice. Find the probability of '*a number greater than 4 on each toss*'.

**Solution** : Let us denote by *A*, the event '*a number greater than 4 on first toss*' and by *B*, the event '*a number greater than 4 on the second toss*'.

Clearly, A and B are independent. ( Why ? )

Thus by Theorem 29,

---

* Some authors call such events dependent.

$$\begin{aligned} P(A \text{ and } B) &= P(A)\,P(B) \\ &= P(5,6)P(5,6) \\ &= \frac{2}{6}\times\frac{2}{6} \\ &= \frac{1}{9} \end{aligned}$$

**Example 7 :** A husband and wife appear in an interview for two vacancies in the same post. The probability of husband's selection is $\frac{1}{7}$ and that of wife's selection is $\frac{1}{5}$. What is the probability that

(a) both of them will be selected ?

(b) only one of them will be selected ?

(c) none of them will be selected ?

**Solution :** We let A : husband is selected.

B : wife is selected.

Clearly, '*A*' and '*B*' are independent. Also '*A*' and '*not B*' are independent ; and '*not A*' are independent ; '*not A*' and '*not B*' are independent.

(a) $P(A \text{ and } B) = P(A)\,P(B)$

$$= \frac{1}{7}\times\frac{1}{5} = \frac{1}{35}$$

(b) P(only one of them will be selected)

$$\begin{aligned} &= P(\text{'A and not B' or 'B and not A'}) \\ &= P(A \text{ and not } B) + P(B \text{ and not } A) \quad \text{(Why ?)} \\ &= P(A)\,P(\text{not } B) + P(B)\,P(\text{not } A) \\ &= P(A)\,[1-P(B)] + P(B)\,[1-P(A)] \\ &= \frac{1}{7}\left(1-\frac{1}{5}\right)+\frac{1}{5}\left(1-\frac{1}{7}\right) \\ &= \frac{1}{7}\times\frac{4}{5}+\frac{1}{5}\times\frac{6}{7} = \frac{10}{35} = \frac{2}{7} \end{aligned}$$

(c) P(none of them will be selected)

$$= \text{P(not A and not B)}$$

$$= \text{P(not A) P(not B)}$$

$$= \frac{6}{7} \times \frac{4}{5} = \frac{24}{35}$$

## EXERCISE 27.2

1. State which of the following events are mutually exclusive. Give reasons for your answer.

   (a) A : a total of 8, B : a total of 12, in a single throw of two dice.

   (b) A : a red card, B : an ace, in a draw of a card from a deck.

   (c) A : a mother, B : a woman, in selecting the President of a ladies club.

   (d) A : owns a house, B : owns a T.V. set, in getting a response from an individual selected in a survey.

   (e) A : a total of 8, B : an even number on one toss in two successive tosses of a die.

   (f) A : 2 heads, B : at least one head, in two successive tosses of a coin.

   (g) A : a total of 11, B : an odd number on each die, in a simultaneous throw of two dice.

2. In a single throw of two dice, find P (a total of 8 or 12).

3. In a simultaneous throw of two dice, find P (A or B) if A denotes the event '*a total of 11*' and B denotes the event '*an odd number on each die*'

4. A card is drawn from a well-shuffled deck of 52 cards. Find the probability of drawing

   (a) a black king

   (b) a jack, queen, king or ace

(c) a card which is neither a heart nor a king

(d) a spade or a club

5. A and B are two events such that P $(A)=0.42$, P $(B)=0.48$ and P (A and B) $= 0.16$. Determine

(a) P (not A)

(b) P (not B)

(c) P (A or B)

6. In a single throw of three dice, determine the probability of getting

(a) a total of 5

*(b) a total of at most 5

*(c) a total of at least 5

7. State which of the following events are independent. Give reasons for your answer.

(a) A : 2 heads, B : 2 tails, in two successive throws of a pair of coins.

(b) A : a black card, B : a red card, in two successive draws of a card without replacement, from a deck.

(c) A : an even number, B : an odd number, in two successive throws of a die.

(d) A : getting a total of 11, B : getting a total of 11, in two successive throws of a pair of dice.

8. In a single throw of two dice, determine the probability of getting

(a) a total of 2

(b) a total of 12

(c) a total of 7 or 9

9. In two successive throws of a pair of dice, determine the probability of getting a total of 8, each time.

10. A coin is tossed successively three times. Determine the probability of

getting

(a) exactly 2 heads

(b) at least 2 heads

(c) at most 2 heads

11. Given two independent events A, B such that P (A)=0.30 and P (B)=0.60. Determine

(a) P (A and B)

(b) P (A and not B)

(c) P (not A and B)

(d) *P* (neither A nor B)

(e) P (A or B)

12. A card is drawn from a well-shuffled deck of 52 cards. The outcome is noted, the card is replaced and the deck reshuffled. Another card is then drawn from the deck. What is the probability that the first card is an ace and the second card is a red queen ?

13. In a single throw of two dice, find the probability that neither a doublet (same number on both the dice) nor a total of 9 will appear.

14. An urn contains 25 balls numbered 1 through 25. Suppose an odd number is considered a 'success.' Two balls are drawn from the urn with replacement. Find the probability of getting

(a) two successes

(b) exactly one success

(c) at least one success

(d) no success

15. An urn contains 7 red and 4 blue balls. Two balls are drawn at random with replacement. Find the probability of getting

(a) 2 red balls

(b) 2 blue balls

(c) one red and one blue ball

**16.** A die is thrown 3 times. Getting a '5 or a 6' is considered a success. Find the probability of

(a) 3 successes

(b) exactly 2 successes

(c) at most 2 successes

(d) at least 2 successes

**17.** Two coins are tossed simultaneously. Complete the following table:

| Number of Heads : | 0 | 1 | 2 |
|---|---|---|---|
| Probability : | | | |

**18.** Three coins are tossed simultaneously. Complete the following table :

| Number of Heads : | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| Probability : | | | | |

**19.** Four coins are tossed simultaneously. Complete the following table :

| Number of Heads : | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| Probability : | | | | | |

**20.** Two dice are tossed simultaneously. Complete the following table :

| Total number of points : | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Probability : | | | | | | | | | | | |

## 27.4 Key Concepts

Sample space

Sample points

Equally likely outcomes

Probability of an event

Probability of an impossible event

Probability of a sure (certain) event

Mutually exclusive events

P(A or B)

P(not A)

Independent events

P(A and B)

## 27.5 Suggestions for Further Reading

*Practically every textbook in Statistics has an introductory chapter on probability.* The reader is referred to

[1] H. L. Alder and E. B. Roessler: **Introduction to Probability and Statistics,** Sixth Edition.
W.H. Freeman and Company, San Francisco (U.S.A.). 1977.

[2] D. H. Sanders, etc.: **Statistics—A Fresh Approach.**
McGraw Hill Book Company, New York (U.S.A.). 1976.

*A set-theoretic approach to basic concepts in probability is given in*

[3] John E. Freund: **Modern Elementary Statistics,** Third Edition.
Prentice-Hall, Inc., New Jersey, (U.S.A.). 1967.

*An excellent treatment of axiomatic approach at rather advanced level, is found in*

[4] A. N. Kolmogorov: **Foundations of Probability,** Second English Edition.
Chelsea Publishing Company, New York (U.S.A.). 1956.

*A detailed history of the theory of Probability from the time of* **Pascal** *to that of* **Laplace** *is given in a reprint of 1865 Cambridge Edition of*

[5] I. Todhunter: **A History of the Mathematical Theory of Probability.**
**Chelsea Publishing Company, New York (U.S.A.). 1949.**

# UNIT XXVIII

# RANDOM VARIABLES AND PROBABILITY DISTRIBUTIONS

*We introduce the notation of a (discrete) random variable and its probability distribution. We learn how to find the mean and variance of a probability distribution.*

## 28.1 Introduction : Discrete Random Variables

Let us return to the origins of probability. A player of a game of chance was not really interested in the probability of each outcome of a game. His main interest was in studying the probability (or chances) of his '*winning*' in a particular game.

Consider, for instance, the game of **rolling a die**. If the rules of the game say that you make money every time you roll, say, *a* '6', your interest would be to know the probability of rolling *a* '6' as against the probability of rolling a 'non-6' (1, 2, 3, 4 or 5) on this die. Rolling *a* '6' is, therefore, a **success** to you.

Or, if you were tossing two coins simultaneously and if the rules of the game say that you make money on the 'number of heads that turn up'. then the occurrence of this event is a **success** to you and naturally, you would be interested in the probability of say '2 successes' or '1 success', etc.

**Getting a 6** in a toss of a die, **the number of heads** in a simultaneous toss of two coins are examples of what we call* **discrete random variables.** Some other examples of random variables are

(i) the number of sixes in a toss of 2 dice,

(ii) the number of aces in a draw of 2 cards from a well-shuffled deck,

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*In this unit, we will study discrete random variables only. We will simply call them random variables.

(iii) the number of black balls drawn from an urn containing 6 red and 4 black balls when, say, 3 balls are drawn from this urn, without replacement,

(iv) the number of tails in 3 tosses of a coin or in a simultaneous toss of 3 coins,

(v) the number of defective bulbs in a lot of 30 items which contains a certain portion of defective bulbs when, say, 4 bulbs are drawn from this lot, etc.

Can you think of some more examples ? *It is intuitively clear that the occurrence of these events involves an element of uncertainty and thus probability enters into the picture.* In the next section, we shall see how.

## 28.2 Probability Distributions

**We denote random variables by capital letters, usually, from the last part of the alphabet, for instance, X, Y, Z, etc.**

Let us consider the simple example of **throwing a die** and let us suppose that **the die is thrown once.** We have already remarked that **'getting a 6'** is considered a **'success'** in the rules of this game and that it is a **random variable.** Let us denote it by X. **Then what are the possible values of X ?**

Clearly X can take the values 0 or 1. [A throw of a die can result in either a 6 **(success)** or a non-six **(failure)**. If it is a 6, X takes the value 1. If it is a non-six, X takes the value 0.]

Assuming that the die is 'fair', what is the probability of each of these events ? It is rather easy to see that

$$P(X=0) = P(\text{non-six}) = \frac{5}{6}$$

$$P(X=1) = P(6) = \frac{1}{6}$$

**Thus X takes the value 0 with probability $\frac{5}{6}$ and the value 1 with probability $\frac{1}{6}$. We write this as follows :**

| X : | 0 | 1 |
|---|---|---|
| $P(X)$ : | $\frac{5}{6}$ | $\frac{1}{6}$ |

We observe that **each of the probabilities is a non-negative fraction (never greater than 1) and that their sum is 1**. We say that the above is the **probability distribution of the random variable X** or simply, the **probability distribution of X.**

We consider some more examples.

**Example 1** A (fair) die is tossed once. If the random variable is 'getting an even number' (denote it by Y), find the probability distribution of Y.

**Solution :** Either we get an even number or we do not get an even number, when we toss a die once. Thus, it is clear that Y takes the values 1 or 0. Now,

$$P(Y=1) = P(2,\ 4 \text{ or } 6) = \frac{1}{2}$$

$$P(Y=0) = P(1,3 \text{ or } 5) = \frac{1}{2}$$

which is the required probability distribution of Y. We could just as well write it in the form of a table as

| Y : | 0 | 1 |
|---|---|---|
| P (Y) : | $\frac{1}{2}$ | $\frac{1}{2}$ |

Again, we observe that **each of the probabilities is a non-negative fraction (never greater than 1) and that their sum is 1.**

**Example 2 :** Find the probability distribution of X, the number of heads, in the two tosses of a coin (or a simultaneous toss of two coins). [See also Q. 17. Exercise 27.2]

**Solution** : What are the possible values of X? Clearly X can take the values 0, 1 or 2

$$\begin{aligned} \text{Now,}\quad P(X=0) &= P(\text{no heads}) \\ &= P(\text{Tails on the first toss and Tails on the second toss}) \\ &= P(\text{Tails on the first toss})\ P(\text{Tails on the second toss}) \\ &= \frac{1}{2} \times \frac{1}{2} = \frac{1}{4} \end{aligned}$$

$$P(X=1)=P(1 \text{ head})= P(HT \text{ or } TH)$$
$$= P(HT) + P(TH)$$
$$= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

$$P(X=2)=P(2 \text{ heads})=P(HH)=\frac{1}{4}$$

Thus, we have

| X : | 0 | 1 | 2 |
|---|---|---|---|
| P(X) : | $\frac{1}{4}$ | $\frac{1}{2}$ | $\frac{1}{4}$ |

Again, we observe that **each of the probabilities is a non-negative fraction (never greater than 1) and that their sum is 1.**

**Remark :** It is clear that the probability distribution of the number of tails (Y) in the two tosses of a coin is the same as above.

**Example 3 :** Find the probability distribution of Z, the number of 'sixes', in two tosses of a die (or a simultaneous toss of two dice).

**Solution :** Clearly the possible values of Z are 0, 1 or 2. Recall that the probability of a 'six' on a toss of a die is $\frac{1}{6}$ and that of a 'non-six' is $\frac{5}{6}$. Thus

$$P(Z=0) = P(\text{non-six and non-six})$$
$$= P(\text{non-six on the first toss})\ P(\text{non-six on the second toss})$$
$$= \frac{5}{6} \times \frac{5}{6} = \frac{25}{36}$$
$$P(Z=1) = P(\text{non-six and a six}) + P(\text{a six and non six}) \qquad \text{(Why ?)}$$
$$= P(\text{non-six})\ P(\text{six}) + P(\text{six})\ P(\text{non-six})$$
$$= \frac{5}{6} \times \frac{1}{6} + \frac{1}{6} \times \frac{5}{6} = \frac{10}{36}$$
$$P(Z=2) = P(\text{ six and six}) = P(\text{ six})\ P(\text{ six}) = \frac{1}{6} \times \frac{1}{6}$$
$$= \frac{1}{36}$$

Thus, we have

| $Z$ : | 0 | 1 | 2 |
|---|---|---|---|
| $P(Z)$ : | $\frac{25}{36}$ | $\frac{10}{36}$ | $\frac{1}{36}$ |

Again, we observe that **each of the probabilities is a non-negative fraction (never greater than 1) and that their sum is 1.**

**Example 4 :** A die is tossed twice. A 'success' is 'getting an odd number' on each toss. Find the probability distribution of the number of successes, S

**Solution :** Clearly S can take the values 0, 1, or 2.

Now, $P(\text{S in one toss}) = P(1,3,5) = \frac{3}{6} = \frac{1}{2}$

$P(\text{not S in the same toss}) = P(2,4,6) = \frac{1}{2}$

Thus, $P(S=0) = P(\text{not S and not–S})$

$$= \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

$$P(S=1) = P(\text{S and not-S}) + P(\text{not-S and S})$$

$$= \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} = \frac{1}{2}$$

$$P\ (S=2) = P\ (\text{S and S}) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

Thus, we have

| S : | 0 | 1 | 2 |
|---|---|---|---|
| P (S) : | $\frac{1}{4}$ | $\frac{1}{2}$ | $\frac{1}{4}$ |

Again, we observe that **each of the probabilities is a non-negative fraction (never greater than 1) and that their sum is 1.**

**Example 5 :** Two cards are drawn successively, with replacement, from a well-shuffled deck of 52 cards. Find the probability distribution of the number of aces.

**Solution :** The number of aces is a random variable. Let us denote it by X. Clearly X takes the values 0, 1 or 2.

$$P\ (X=2)=P\ (\text{ace and an ace})=P\ (\text{ace})\ P\ (\text{ace})$$

$$=\frac{4}{52}\times\frac{4}{52}=\frac{1}{169}$$

$$P\ (X=1)=P\ (\text{ace and a non-ace})+P\ (\text{non-ace and an ace})$$

$$=\frac{4}{52}\times\frac{48}{52}+\frac{48}{52}\times\frac{4}{52}=\frac{24}{169}$$

$$P\ (X=0)=P\ (\text{non-ace and a non-ace})$$

$$=\frac{48}{52}\times\frac{48}{52}=\frac{144}{169}$$

Thus, we have

| X : | 0 | 1 | 2 |
|---|---|---|---|
| P (X) : | $\frac{144}{169}$ | $\frac{24}{169}$ | $\frac{1}{169}$ |

Again, we observe that **each of the probabilities is a non-negative fraction (never greater than 1) and that their sum is 1.**

**Example 6 :** Two cards are drawn simultaneously (or successively, without replacement) from a well-shuffled deck of 52 cards. Find the probability distribution of the number of aces.

**Solution :** We again let X denote the number of aces. X takes a value 0, 1 or 2.

$$\text{Now, } P\ (X=2)=P\ (\text{both aces})=\frac{C\ (4,2)}{C\ (52,2)}$$

$$=\frac{4\times 3}{52\times 51}=\frac{1}{221}$$

$$P(X=1) = P\text{ (one ace)} = P\text{(one ace and one non-ace)}$$

$$= \frac{C(4,1)\times C(48,1)}{C\ (52,2)}$$

$$= 2\left(\frac{4\times 48}{52\times 51}\right) = \frac{32}{221}$$

$$P\ (X=0) = P\text{ (both non-aces)} = \frac{C(48,2)}{C(52,2)}$$

$$= \frac{48\times 47}{52\times 51} = \frac{188}{221}$$

Thus, we have

| X : | 0 | 1 | 2 |
|---|---|---|---|
| P (X) : | $\frac{188}{221}$ | $\frac{32}{221}$ | $\frac{1}{221}$ |

Again, we observe that **each of the probabilities is a non-negative fraction (never greater than 1) and that their sum is 1.**

***Example 7 :** Five defective bulbs are accidentally mixed with twenty **good ones.** It is not possible to just look at a bulb and tell whether or not it is **defective. Find the** probability distribution of the number of defective bulbs, if four **bulbs are drawn at** random from this lot.

**Solution :** Let us denote by X, the number of defective bulbs. **Clearly X can** take a value 0, 1, 2, 3 or 4.

Now, P (X=0)=P (no defective bulbs)=P (all 4 good ones)

$$= \frac{C\ (20,4)}{C\ (25,4)} = \frac{20\times 19\times 18\times 17}{25\times 24\times 23\times 22} = \frac{969}{2530}$$

P (X=1)=P (1 defective and 3 good ones)

$$= \frac{C\ (5,1)\times C\ (20,3)}{C\ (25,4)} = \frac{1140}{2530}$$

P (X=2)=P (2 defectives and 2 good ones)

$$= \frac{C\ (5,2)\times C\ (20,2)}{C(25,4)} = \frac{380}{2530}$$

$$P(X=3) = P(\text{3 defective and 1 good one}) = \frac{C(5,3) \times C(20,1)}{C(25,4)} = \frac{40}{2530}$$

$$P(X=4) = P(\text{all 4 defective}) = \frac{C(5,4)}{C(25,4)} = \frac{1}{2530}$$

Thus, we have

| X : | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| P(X) : | $\frac{969}{2530}$ | $\frac{1140}{2530}$ | $\frac{380}{2530}$ | $\frac{40}{2530}$ | $\frac{1}{2530}$ |

Again, we observe that **each of the probabilities is a non-negative fraction (never greater than 1) and that their sum is 1**.

We see that the **probability distribution of a random variable X spells out how a total probability of 1 is distributed over several values of the random variable.** It is instructive to represent the probability distributions graphically. We represent the values of the random variable on the $x$-axis and probabilities on the $y$-axis. *At each value of the random variable, we draw a segment, parallel to the y-axis, with its length equal to the corresponding probability*. We draw below graphs of some of the probability distributions that we have considered in our examples.

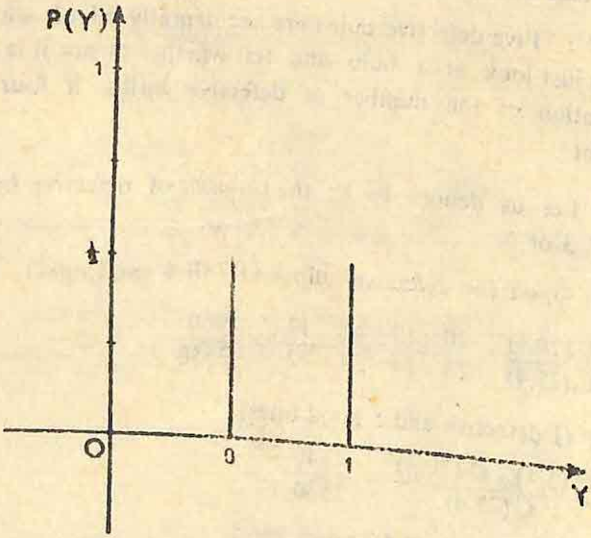

Fig. 28.1 : Graph of the probability distribution of Y in Example 1, Section 28.2

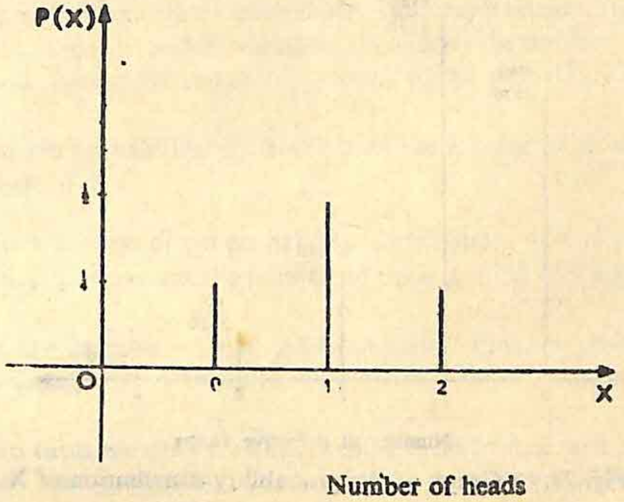

Fig. 28.2 : Graph of the probability distribution of the number of heads in two tosses of a coin (Example 2, Section 28.2)

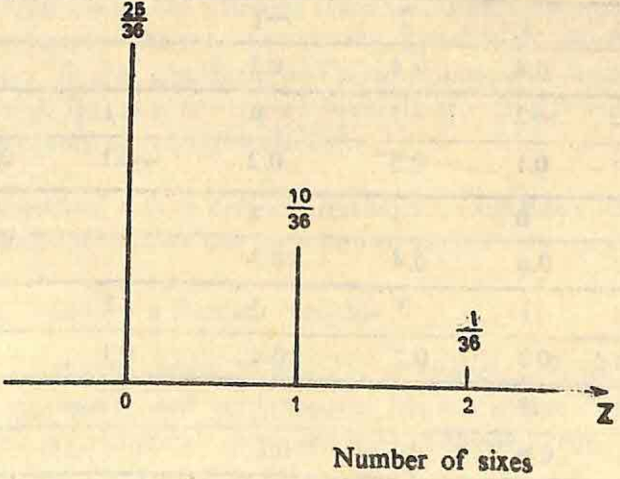

Fig. 28.3 : Graph of the probability distribution of the number of sixes in two throws of a die (Example 3, Section 28.2)

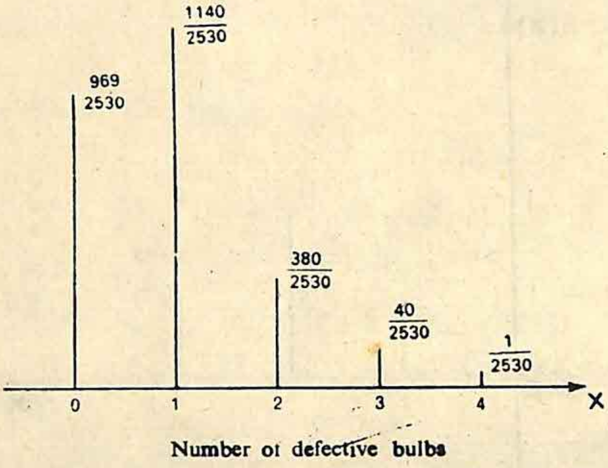

Fig. 28.4 : Graph of the probability distribution of X
in Example 7, Section 28.2

## EXERCISE 28.1

1. Give three examples of discrete random variables.

2. State when it is not possible for a random variable to have any of the following probability distributions. Give reasons for your answer :

| | | | | | | |
|---|---|---|---|---|---|---|
| (a) | X : | 1 | 2 | −1 | | |
| | P(X) : | 0.4 | 0 4 | 0.2 | | |
| (b) | X : | −2 | −1 | 0 | 1 | 2 |
| | P(X) : | 0.1 | 0.5 | 0.2 | −0.1 | 0.3 |
| (c) | Y : | 0 | 6 | 8 | | |
| | P(Y) : | 0.6 | 0.4 | 0.2 | | |
| (d) | Z : | 1 | 3 | 5 | 7 | |
| | P(Z) : | 0.3 | 0.2 | 0.4 | 0.1 | |
| (e) | U : | 18 | 13 | 1 | | |
| | P(U) : | 0.6 | 0.1 | 0.2 | | |

3. Find the probability distribution of the number of successes in two tosses of a die, where a success is defined as 'a number greater than 4'. Sketch its graph.

4. Find the probability distribution of the number of sixes in three tosses of a die and sketch its graph.

5. Three cards are drawn successively, with replacement, from a well-shuffled deck of 52 cards. A random variable X denotes the number of spades in the three cards. Determine and sketch a graph of the probability distribution of X.

6. Find the probability distribution of the number of doublets in four throws of a pair of dice.

7. Sketch a graph of the probability distribution of Y in two throws of two dice, where Y represents the number of times a total of 9 appears.

8. An urn contains 4 white and 3 red balls. Find the probability distribution of the number of red balls in three draws, with replacement, from the urn.

9. Two cards are drawn without replacement from a well shuffled deck. Determine the probability distribution of the number of face cards (Jack, Queen, King and Ace).

10. Two bad eggs are mixed accidentally with 10 good ones. Find the probability distribution of the number of bad eggs in 3,drawn at random, without replacement, from this lot. Sketch a graph of the probability distribution.

### 28.3 Mean and Variance of Random Variables

The reader should see a striking resemblance between probability distributions and frequency distributions. As remarked already, *a probability distribution spells out how a total probability of 1 is distributed over several values of a random variable ; while a frequency distribution spells out how a total frequency of n is distributed over several classes (i.e., over class-marks of respective classes)*.

It is, therefore, easy to devise formulae for computing means and variances of probability distributions. We first learn how to calculate the mean.

#### 28.3.1 Mean of a Random Variable

We recall that in finding the mean of a frequency distribution, we summed the products of class-marks and corresponding frequencies and divided the sum by the total of frequencies. We accordingly define the **mean of the probability distribution of a random** variable. *We denote the mean by* $\mu$, *values of* X *by* $x_1, x_2, x_3$, *etc. and the corresponding probabilities, namely,* $P(X=x_1)$ *by* $p_1$, $P(X=x_2)$ *by* $p_2$, $P(X-x_3)$ *by* $p_3$, *etc.*

We define

$$\mu = \frac{\Sigma x_i p_i}{\Sigma p_i} = \Sigma x_i p_i$$

We consider some examples.

**Example 1:** Find the mean of the number of heads in the two tosses of a coin.

**Solution:** We refer to Example 2, Section 28.2, and have the required probability distribution of X. We construct a column of $x_i p_i$'s and show work below:

| $x_i$ | $p_i$ | $x_i p_i$ |
|---|---|---|
| 0 | $\frac{1}{4}$ | 0 |
| 1 | $\frac{1}{2}$ | $\frac{1}{2}$ |
| 2 | $\frac{1}{4}$ | $\frac{2}{4}$ |
| | | 1 |

$\mu = \Sigma x_i p_i = 1$

Thus the mean of the number of heads is 1.

What does it mean? It means that **if we repeat this experiment of two tosses of a coin, 'a large number of times', then, 'on an average', we expect the number of heads to be 1 in each performance of the experiment.** It is not possible to say what we mean by 'a large number of times'. It could be 10 times, 50 times, 1000 times, 20001 times, and so on. All we are saying is that if we record the number of heads each time we toss two coins and we do so for a large number of times, the average of these numbers should be 1.

**Remark:** Other names for the **mean** are **average** or **expected value.** Also the terminology **mean of a random variable** and the **mean of the probability distribution of a random variable** will be interchangeably used.

**Example 2:** Find the average number of sixes in 2 tosses of a die.

**Solution:** Let the number of sixes be denoted by Z.

The required probability distribution is given in Example 3, Section 28.2. We carry out the computation of the mean as below :

| $z_i$ | $p_i$ | $z_ip_i$ |
|---|---|---|
| 0 | $\frac{25}{36}$ | 0 |
| 1 | $\frac{10}{36}$ | $\frac{10}{36}$ |
| 2 | $\frac{1}{36}$ | $\frac{2}{36}$ |

We see that $\Sigma\, z_ip_i = \frac{12}{36} = \frac{1}{3}$

Thus, the average number of sixes is $\frac{1}{3}$.

**Example 3 :** Find the mean number of heads in three tosses of a coin (or a simultaneous toss of three coins).

**Solution :** Let us denote by $Y$ the number of heads. We refer to Q. 18, Ex. 27.2 to obtain the required probability distribution of $Y$. We carry out the computation of the mean as below :

| $y_i$ | $p_i$ | $y_ip_i$ |
|---|---|---|
| 0 | $\frac{1}{8}$ | 0 |
| 1 | $\frac{3}{8}$ | $\frac{3}{8}$ |
| 2 | $\frac{3}{8}$ | $\frac{6}{8}$ |
| 3 | $\frac{1}{8}$ | $\frac{3}{8}$ |

We see that $\Sigma\, y_ip_i = \frac{12}{8} = \frac{3}{2}$

Thus, the mean number of heads is $\frac{3}{2}$.

**Example 4** : Find the mean of the following probability distribution

| $X$ : | $-3$ | $-1$ | $0$ | $4$ |
|---|---|---|---|---|
| $P(X)$ : | 0.2 | 0.4 | 0.3 | 0.1 |

**Solution** : We carry out the computation of the mean as below :

| $x_i$ | $p_i$ | $x_ip_i$ |
|---|---|---|
| $-3$ | 0.2 | $-0.6$ |
| $-1$ | 0.4 | $-0.4$ |
| 0 | 0.3 | 0 |
| 4 | 0.1 | 0.4 |

We see that $\Sigma x_ip_i = -0.6$

Thus, the mean of the given probability distribution is $-0.6$

### 28.3.2 Variance of a Random Variable

As before, we get a clue from the formula for variance of a frequency distribution to write a formula for calculation of the variance of the probability distribution of a random variable. We denote the variance by $\sigma^2$. We have

$$\sigma^2 = \Sigma(x_i - \mu)^2 p_i \tag{1}$$

If, however, $\mu$ is not a 'nice' number, we have the following formula for $\sigma^2$ :

$$\sigma^2 = \Sigma x^2_i p_i - \mu^2 \tag{2}$$

**Standard deviation** $\sigma$, **is, of course,** $+\sqrt{\sigma^2}$

We consider some examples.

**Example 5** : Find the variance of the number of heads in the two tosses of a coin. (See also Example 1, Section 28.3.1)

**Solution :** We recall that $\mu=1$. We, therefore, use formula (1) for finding $\sigma^2$ as below :

| $x_i$ | $p_i$ | $x_i-\mu$ | $(x_i-\mu)^2$ | $(x_i-\mu)^2p_i$ |
|---|---|---|---|---|
| 0 | $\frac{1}{4}$ | $0-1$ | 1 | $1\times\frac{1}{4}=\frac{1}{4}$ |
| 1 | $\frac{1}{2}$ | $1-1$ | 0 | $0\times\frac{1}{2}=0$ |
| 2 | $\frac{1}{4}$ | $2-1$ | 1 | $1\times\frac{1}{4}=\frac{1}{4}$ |

Thus, $\sigma^2=\Sigma\,(x_i-\mu)^2\,p_i=\frac{1}{2}$

**Example 6 :** Determine the standard deviation and variance of the number of sixes in 2 tosses of a die. (See also Example 2, Section 28.3.1)

**Solution :** We recall that $\mu=\frac{1}{3}$. We, therefore, use formula (2) for finding $\sigma^2$ and hence, $\sigma$. We have

| $z_i$ | $p_i$ | $z_i^2$ | $z_i^2p_i$ |
|---|---|---|---|
| 0 | $\frac{25}{36}$ | 0 | $0\times\frac{25}{36}=0$ |
| 1 | $\frac{10}{36}$ | 1 | $1\times\frac{10}{36}=\frac{10}{36}$ |
| 2 | $\frac{1}{36}$ | 4 | $4\times\frac{1}{36}=\frac{4}{36}$ |

We note that $\Sigma\,z_i^2p_i=\frac{14}{36}=\frac{7}{18}$

Thus, $\sigma^2=\Sigma z_i^2p_i-(\mu)^2$

$$=\frac{7}{18}-\left(\frac{1}{3}\right)^2=\frac{7}{18}-\frac{1}{9}=\frac{5}{18}$$

And, $\sigma=+\sqrt{\sigma^2}=+\sqrt{\frac{5}{18}}=0.527$ (approx.)

**Example 7 :** Find the variance of the number of heads in three tosses of a coin. (See also Example 3, Section 28.3.1)

**Solution :** We recall that $\mu = \frac{3}{2}$. We again use (2) to find $\sigma^2$.

| $y_i$ | $p_i$ | $y_i^2$ | $y_i^2 p_i$ |
|---|---|---|---|
| 0 | $\frac{1}{8}$ | 0 | $0 \times \frac{1}{8} = 0$ |
| 1 | $\frac{3}{8}$ | 1 | $1 \times \frac{3}{8} = \frac{3}{8}$ |
| 2 | $\frac{3}{8}$ | 4 | $4 \times \frac{3}{8} = \frac{12}{8}$ |
| 3 | $\frac{1}{8}$ | 9 | $9 \times \frac{1}{8} = \frac{9}{8}$ |

We see that $\Sigma\, y_i^2 p_i = \frac{24}{8} = 3$

Thus, $\sigma^2 = \Sigma y_i^2 p_i - (\mu)^2 = 3 - \left(\frac{3}{2}\right)^2 = \frac{3}{4}$

**Example 8 :** Compute $\sigma^2$ and $\sigma$ for the following probability distribution :

| X : | $-3$ | $-1$ | 0 | 4 |
|---|---|---|---|---|
| P(X) : | 0.2 | 0.4 | 0.3 | 0.1 |

(See also Example 4, Section 28.3.1)

**Solution :** We recall that $\mu = -0.6$. We use formula (2) and have the following calculations for $\sigma^2$ :

| $x_i$ | $p_i$ | $x_i^2$ | $x_i^2 p_i$ |
|---|---|---|---|
| $-3$ | 0.2 | 9 | $9(0.2) = 1.8$ |
| $-1$ | 0.4 | 1 | $1(0.4) = 0.4$ |
| 0 | 0.3 | 0 | $0(0.3) = 0$ |
| 4 | 0.1 | 16 | $16(0.1) = 1.6$ |
| | | | 3.8 |

$$\begin{aligned} \text{Thus, } \sigma^2 &= \Sigma x_i^2 p_i - (\mu)^2 \\ &= 3.8 - (-0.6)^2 \\ &= 3.44 \end{aligned}$$

And, $\sigma = +\sqrt{3.44} = 1.85$ (approx.)

## EXERCISE 28.2

Find $\mu$, $\sigma^2$ and $\sigma$ for each of the following probability distributions :

1.

| $X$ : | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $P(X)$ : | 0.4 | 0.3 | 0.2 | 0.1 |

2.

| $Y$ : | $-2$ | $-1$ | 0 | 1 | 2 |
|---|---|---|---|---|---|
| $P(Y)$ : | 0.1 | 0.2 | 0.4 | 0.2 | 0.1 |

3.

| $X$ : | 0 | 1 | 3 | 5 |
|---|---|---|---|---|
| $P(X)$ : | 0.2 | 0.5 | 0.2 | 0.1 |

4

| $Z$ : | $-3$ | $-1$ | 0 | 1 | 3 |
|---|---|---|---|---|---|
| $P(Z)$ : | 0.05 | 0.45 | 0.20 | 0.25 | 0.05 |

5.

| $Z$ : | $-2$ | $-1$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|
| $P(Z)$ : | 0.1 | 0.1 | 0.2 | 0.1 | 0.2 | 0.3 |

6. A die is tossed twice. A 'success' is 'getting an odd number' on a toss. Find the variance of the number of successes.

7. Two cards are drawn successively, with replacement, from a well-shuffled deck of 52 cards. Find $\mu$ and $\sigma$ for the number of aces.

8. Two cards are drawn simultaneously (or successively, without replacement) from a well-shuffled deck of 52 cards. Compute $\sigma^2$ for the number of aces.

9. Five defective bulbs are accidentally mixed with twenty good ones. It is not possible to just look at a bulb and tell whether or not a bulb is defective. Four bulbs are drawn at random from this lot. Find the mean number of defective bulbs drawn.

10. A die is tossed twice. Getting 'a number greater than 4' is considered a 'success'. Find the variance of the probability distribution of the number of successes.

## 28.4 Key Concepts

Discrete random variable

Probability distribution of a random variable

Mean of the probability distribution of a random variable

Variance of the probability distribution of a random variable

Standard deviation of the probability distribution of a random variable

## 28.5 Suggestions for Further Reading

*Again, any standard book on Probability will deal with probability distributions. The reader is referred, for instance, to* [1] *and* [2], *Section 27.5.*

*An excellent paper back is*

[1] M. Willerding : **A Probability Primer**.

. Prindle, Weber and Schmidt, Inc., Boston, (U.S.A.). 1968.

## MISCELLANEOUS EXERCISE VII

### (On Units XXVII and XXVIII )

1. Three coins are tossed simultaneously. List all the possible outcomes. What is the probability of 'exactly 3 heads', 'exactly 2 tails' and 'at most 2 heads' ?

2. What is the probability of drawing 'a red card' from a well-shuffled deck of 52 cards ? What is the probability of drawing 'a black ace' ?

3. A bag contains 3 black and 5 white marbles. A marble is drawn 'at random' from the bag. What is the probability that the marble drawn is 'not white' ?

4. In a single throw of two dice, find the probability that the total number of dots obtained is a multiple of 3.

5. A die has two faces each with number '1', three faces each with number '2' and one face with number '3'. If the die is rolled once, determine
   (i) P(2)
   (ii) P(1 or 3)
   (iii) P(not 3)

6. In a single throw of two dice, find the probability of a total of an odd number greater than 5.

7. The probabilities that a student will receive an A, B, C or D grade are 0.40, 0.35, 0.15 and 0.10 respectively. Find the probability that a student will receive
   (i) not an A grade
   (ii) at most a C grade
   (iii) B or C grade

8. Two dice are tossed once. Find the probability of getting 'a multiple of 3 on the first die or a total of 10'

9. What is the probability of not getting a 1 or 6 in a single toss of a die ?

10. Given two mutually exclusive events A and B, such that P(A)=0.45 and P(B)=0.35, find

    (a) P(not A)

    (b) P(not B)

    (c) P(A and B)

11. Find the probability that the sum of the numbers obtained in a simultaneous throw of two dice is an odd number.

12. A bag contains 2 white, 3 black and 4 red balls. A ball is drawn 'at random'. Find the probability that the ball drawn is

    (a) either red or white

    (b) not white

13. A box contains 6 red marbles numbered from 1 through 6 and 4 white marbles numbered from 12 through 15. Find the probability that, a marble drawn 'at random' is

    (a) white

    (b) even numbered

    (c) white and odd numbered

    (d) red or even numbered

14. A die is tossed thrice. Find the probability of 'a number less than 5 on each toss.'

15. In a single throw of three dice, determine the probability of getting a 'triplet' (same number on all the three dice).

16. Given two independent events A and B such that P(A)=0.40, P(B)=0.50, determine

    (a) P(A and B)

    (b) P(A and not B )

(c) P(not A and B)

(d) P(neither A nor B)

(e) P(A or B)

17! A bag contains 13 balls numbered from 1 through 13. Suppose an even number is considered a 'success'. Two balls are drawn, with replacement, from the bag. Find the probability of getting

(a) two successes

(b) exactly one success

(c) at least one success

(d) no success

18. In two successive throws of a pair of dice, determine the probability of getting a total of odd number of points each time.

19. A card is drawn from a well-shuffled deck of 52 cards. The outcome is noted, the card is replaced and the deck is reshuffled. Another card is drawn from the deck. What is the probability that

(a) both the cards are of the same suit ?

(b) both the cards are aces ?

(c) the first card is a spade and the second card is a black king ?

(d) both are face cards ?

*20. Three cards are drawn from a well-shuffled deck of cards, one after the other and with replacement. What is the probability that

(a) all the three cards are spades ?

(b) first two cards are queens and the third card is black ace ?

(c) first card is a jack, second is a red queen and the third card is a king ?

21. A draws two cards with replacement from a well-shuffled deck of cards and at the same time B throws a pair of dice. What is the probability that

(a) A gets both cards of the same suit and B gets a total of 6 ?

(b) A gets two jacks and B gets a doublet.

22. Three persons work independently to decipher a message in Morse Code. The respective probabilities of their deciphering the code are $\frac{1}{4}$, $\frac{1}{3}$ and $\frac{1}{5}$.

What is the probability that the message will be deciphered.

[Hint : the message will be deciphered if any one of the three persons can decipher it. ]

23. What is the probability of throwing at least two sevens in 3 throws of a pair of dice ?

24. If 1 out of 10 bulbs produced in a factory is defective, what is the probability that out of a sample of 3 bulbs, with replacement,

(a) exactly 2 bulbs are defective ?

(b) not more than 2 bulbs are defective ?

(c) no bulb is defective ?

25. In a simultaneous toss of 4 coins, what is the probability of getting

(a) less than 3 heads ?

(b) more than 3 heads ?

(c) exactly 3 heads ?

26. Find $\mu$, $\sigma^2$, $\sigma$ for each of the following probability distributions :

(a)

| X : | 1 | 2 | 3 | −2 | −1 |
|---|---|---|---|---|---|
| P(X) : | 0.10 | 0.35 | 0.20 | 0.30 | 0.05 |

(b)

| X : | 1 | 3 | 5 | 7 | 9 |
|---|---|---|---|---|---|
| P(X) : | 0.1 | 0.0 | 0.4 | 0.2 | 0.3 |

(c)

| Z : | 0 | 2 | 4 | −1 | 3 |
|---|---|---|---|---|---|
| P(Z) : | 0.05 | 0.25 | 0.20 | 0.45 | 0.05 |

27. A die is tossed thrice. A 'success' is 'getting 1 or 6' on a toss. Find the mean and the variance of the number of successes.

28. Compute the variance of the probability distribution of the number of doublets in four throws of a pair of dice.

29. An urn contains 4 white and 3 red balls. Three balls are drawn, with replacement, from this urn. Find $\mu$, $\sigma^2$ and $\sigma$ for the number of red balls drawn.

*30. Two bad eggs are accidentally mixed with ten good ones. Three eggs are drawn at random, without replacement, from this lot. Compute $\mu$ for the number of bad eggs drawn.

# UNIT XXIX

## THE PROBLEM OF AREAS

### (An Introduction to the Integral)

*In this unit, we, study* **the problem of areas** *as a motivation for the integral. We learn some properties of the integral and use the* **Fundamental Theorem of Integral Calculus** *to evaluate definite integrals.*

### 29.1 Introduction : The Problem of Areas

We have already studied three important problems that led to the development of Calculus, namely, **the problem of speed, the problem of tangents** and **the problem of maxima and minima.** We now study another important problem which also played a significant role in the development of the concepts in Calculus. This is **the problem of areas** which plagued the early Greek geometers, notably, **Euclid (approx. 300 B. C). and Archimedes (287-212 B. C.).**

We consider some examples.

**Example 1 :** Let us consider the function

$$f(x)=3x+1$$

Recall that its graph is a line. Let us find *the area of the region above the x-axis, below the graph of this function and between the vertical lines at, say, $x=2$ and at another point say, $x>2$.* The region is shaded in Fig. 29.1.

This area is rather easy to compute, since, the region ABDC is a trapezium. Do you recall the area of a trapezium ?

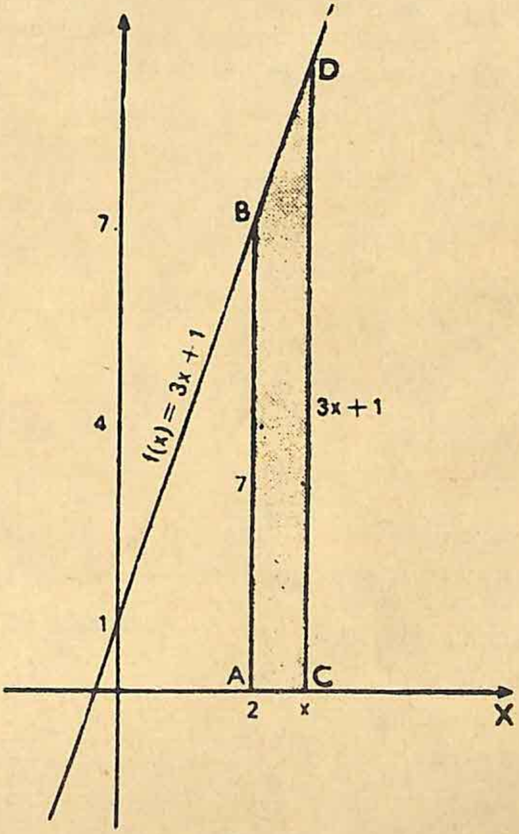

Fig. 29.1

We have

$$\text{ar (ABDC)} = \frac{1}{2}\,(\,\text{AB} + \text{CD}\,) \times \text{AC}$$

$$= \frac{1}{2}\,(7 + 3x + 1)\,(x - 2)$$

i.e., $$\text{ar (ABDC)} = \frac{1}{2}\,(3x + 8)\,(x - 2)$$

We note that this area is a function of x. If we denote the ar (ABDC) by $A\,(x)$, we have

$$A\,(x) = \frac{3}{2}\,x^2 + x - 8 \qquad (1)$$

Thus, from $f(x)$, we have obtained a new function $A(x)$ defined for $x>2$, it represents the area above the $x$-axis bounded by the graph of $f(x)$ between the vertical lines at $x=2$ and at another point $x>2$. We call $A(x)$ **the area function.** We say **briefly that A (x) is the area under f (x) between the ordinates at 2 and x.**

Let us now find the *derivative of the area function with respect to x*. We have

$$A'(x)=\frac{3}{2}(2x)+1=3x+1$$

What do we observe? **We observe that**

$$A'(x)=f(x), \text{ for } x>2$$

**i.e., the derivative of the area function A (x) is the given function f (x)** in the interval $x>2$.

**Remark 1 :** It is intuitively clear that *the area under $f(x)$, above the x-axis and between the ordinates at x=2 and 2 (again) is zero*. We also see that $\frac{3}{2}x^2+x-8$ has value zero for $x=2$. We can, therefore, extend the **area function A (x) and have**

$$A(x)=\frac{3}{2}x^2+x-8, \text{ for } x\geqslant 2$$

We also see that

$$A'(2)=f(2)$$

**Remark 2 :** What if we were to find the area under the function $f(x)=3x+1$ between the ordinates at, say, $x=3$ (instead of $x=2$) and at some $x\geqslant 3$ ?

The reader is advised to compute this area, say, $A(x)$ by using, again, the formula for the area of a trapezium and show that

$$A(x)=\frac{3}{2}x^2+x-\frac{33}{2}, \text{ for } x\geqslant 3$$

What is $A(3)$ ?

What is $A'(x)$ ?

We again observe that the derivative of the area function A (x) is the given function f (x) for $x>3$.

Example 2 : Let us consider the function

$$g(x) = -2x+3$$

Recall that its graph is a line

Let us find *the area* A $(x)$ *under* $g(x)$ *between the ordinates at, say,* $x=1$ *and at some* $x \geqslant 1$. The region in question is shaded in Fig. 29.2.

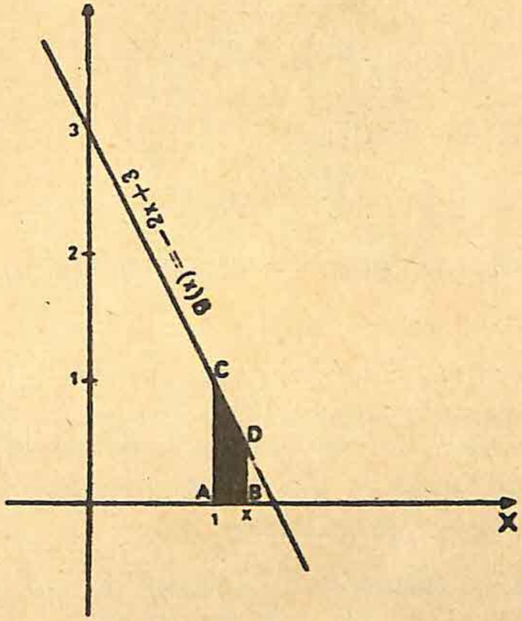

Fig. 29.2

We have

$$A(x) = \frac{1}{2}(1-2x+3)(x-1)$$

$$\text{or, } A(x) = -x^2+3x-2$$

What is $A'(x)$ ? We have

$$A'(x) = -2x+3 = g(x) \quad \text{for } x \geqslant 1.$$

We again observe that **the derivative of the area function A (x) is the given function g $(x)$ for x$\geqslant$1**.

**Remark 1 : Let us find A (1).** We have

$$A(1) = -(1)^2 + 3(1) - 2 = \mathbf{0},$$

as was expected.

**Remark 2 :** Let us now find the area A $(x)$ under the same function $g(x)$ between the ordinates at, say, $x=0$ and at some $x \geqslant 0$. We have

$$A(x) = \frac{1}{2}(3 - 2x + 3)x = 3x - x^2$$

What is A (0) ?

What is A$'$ $(x)$ ?

We again, observe that **the derivative of the area function** A(x) **is the given function g (x) for x $\geqslant$0**.

The area functions A $(x)$, in Examples 1 and 2 above, were rather easy to compute using the methods of Elementary Geometry. This is because the graph of $f(x)$ in both the examples is a line. Euclidean Geometry provides us methods of finding areas of even more complicated rectilinear figures by '*decomposing*' them into triangles. Recall, for instance, how we find the area of a regular polygon.

**The first really difficult problem of area is finding the area of a circular region.** The Greeks wanted to **square a circle** in the sense that they attempted to construct, by rular and compass, a square whose area would be equal to area of a given circle. That this is impossible was proved in 1882 by **Ferdinand Lindemann (1852-1939 A.D.)** However, the Greek mathematician **Archimedes** was successful in **squaring a region bounded by the arc of a parabola**. In the next section, we consider the problem of finding the area of a region bounded by an arc of a parabola between the ordinates at $x=0$ and some point $x$.

## 29.2 Squaring the Parabola

Let the parabola be given by the equation $y=x^2$. We wish to find the area A $(x)$ of the shaded region. (See Fig. 29.3) *Can you, from your earlier classes, recall some*

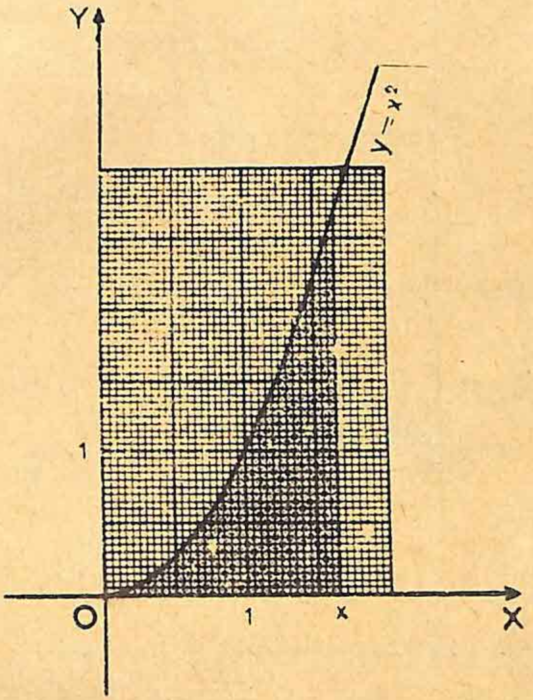

Fig. 29.3 : Area by the Grid Method

*method of finding this area*? Yes. We could, for instance, cover the plane with a grid of squares and count the number of squares lying wholly within the shaded area. But as we shall see, we will always have some squares which will lie partly within and partly outside the shaded area. **We can, therefore, only get an approximation to the area by this method**. *However, the finer the grid (the smaller the squares), the better will be the approximation to the area.*

We now study the method as proposed by **Archimedes**. We divide the interval from 0 to x into $n$ equal segments, where $n$ is a **positive integer**.

What Archimedes did was to **'sandwich'** $A(x)$ **between** the sum of the areas of rectangles lying completely inside the region **and the sum** of the areas of the rectangles lying partially outside the region as shown in Fig. 29.4.

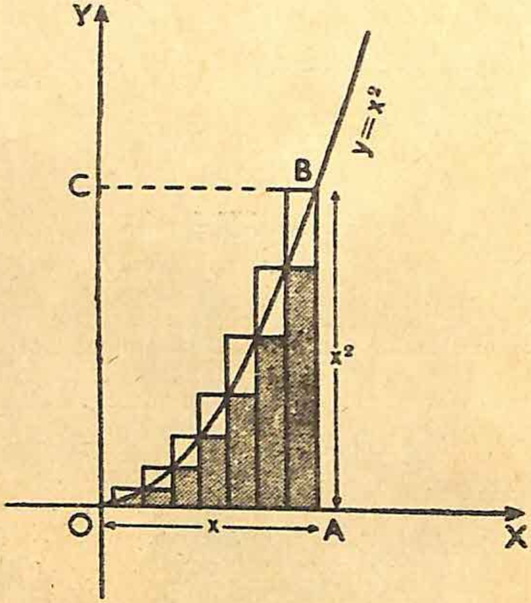

Fig. 29.4

The points of sub-division of the interval from 0 to $x$ are

$$0,\ 0+\frac{x}{n},\ 0+\frac{2x}{n},\ 0+\frac{3x}{n},\ \dots,\ 0+\frac{(n-1)x}{n},\ 0+\frac{nx}{n}$$

Thus, the length of each segment is $\frac{x}{n}$. What is the ordinate at each point of the sub-division? The ordinate at $\frac{x}{n}$ is $\left(\frac{x}{n}\right)^2$, at $\frac{2x}{n}$ is $\left(\frac{2x}{n}\right)^2$, etc.

Now what is the sum of the areas of rectangles lying partially outside the shaded region? We have

$$\frac{x}{n}\left(\frac{x}{n}\right)^2+\frac{x}{n}\left(\frac{2x}{n}\right)^2+\frac{x}{n}\left(\frac{3x}{n}\right)^2+\dots+\frac{x}{n}\left(\frac{(n-1)x}{n}\right)^2+\frac{x}{n}\left(\frac{nx}{n}\right)^2$$

$$=\frac{x^3}{n^3}\left[1^2+2^2+3^2+\dots+(n-1)^2+n^2\right]$$

$$=\frac{x^3}{n^3}\,\frac{n(n+1)(2n+1)}{6}=\frac{x^3}{6}\left(\frac{2n^2+3n+1}{n^2}\right)$$

$$= \frac{x^3}{6}\left(2+\frac{3}{n}+\frac{1}{n^2}\right)$$

$$= \frac{x^3}{3}+\frac{x^3}{2n}+\frac{x^3}{6n^2} \qquad (1)$$

What about the sum of the areas of the rectangles lying completely inside the shaded region? We have

$$\frac{x}{n}(0)^2+\frac{x}{n}\left(\frac{x}{n}\right)^2+\frac{x}{n}\left(\frac{2x}{n}\right)^2+\cdots+\frac{x}{n}\left(\frac{(n-1)x}{n}\right)^2$$

$$= \frac{x^3}{n^3}\left[0^2+1^2+2^2+\cdots+(n-1)^2\right]$$

$$= \frac{x^3}{n^3}\,\frac{(n-1)n(2n-1)}{6} = \frac{x^3}{6}\left(\frac{2n^2-3n+1}{n^2}\right)$$

$$= \frac{x^3}{3}-\frac{x^3}{2n}+\frac{x^3}{6n^2} \qquad (2)$$

Now Archimedes says, it is intuitively clear that

$$(2) \quad < A(x) < (1)$$

$$\text{i.e., } \frac{x^3}{3}-\frac{x^3}{2n}+\frac{x^3}{6n^2} < A(x) < \frac{x^3}{3}+\frac{x^3}{2n}+\frac{x^3}{6n^2} \qquad (3)$$

Dividing throughout by $x^3$ (which is the area of the rectangle OABC and hence positive), we rewrite (3) as

$$\frac{1}{3}-\frac{1}{2n}+\frac{1}{6n^2} < \frac{A(x)}{x^3} < \frac{1}{3}+\frac{1}{2n}+\frac{1}{6n^2} \qquad (4)^*$$

---

$$^*\frac{1}{2n}+\frac{1}{6n^2} = \frac{3n+1}{6n^2} < \frac{3n+3n}{6n^2} = \frac{1}{n}$$

$$-\frac{1}{2n}+\frac{1}{6n^2} = \frac{-3n+1}{6n^2} > \frac{-3n-3n}{6n^2} = -\frac{1}{n}$$

Thus (4) can be written as

$$\frac{1}{3}-\frac{1}{n} < \frac{A(x)}{x^3} < \frac{1}{3}+\frac{1}{n}$$

Archimedes then argued that as $n$, the number of subdivisions, becomes large,

$\frac{A(x)}{x^3} = \frac{1}{3}$ by what we call the **Sandwich Theorem**

$$\text{i.e., } A(x) = \frac{x^3}{3}$$

Archimedes stated this result in the form of the following theorem :

**Theorem 30 : Let a rectangle be such that one of the vertices is at the vertex of a parabola and its opposite vertex is a point on the parabola in such a way that one of its sides lies on the axis of the parabola. Then the area of this rectangle is divided by the parabola in the ratio 2 : 1.**

We see that $A(0) = 0 = \frac{0^3}{3}$

**What about $A'(x)$ ?** We have

$$A'(x) = \frac{1}{3}(3x^2) = x^2$$

Thus, **the derivative of the area function $A(x)$ is the given function $f(x) = x^2$ for $x \geqslant 0$.**

**Remark** : How shall we find the area under $f(x) = x^2$ between the ordinates at, say, $x = 1$ (instead of $x = 0$) and at some other point $x > 1$ ? Geometrically we see that this area $A(x)$ can be written as

$$A(x) = \text{Area between 0 and } x - \text{Area between 0 and 1}$$

$$= \frac{x^3}{3} - \frac{1^3}{3} = \frac{x^3 - 1}{3}$$

We see that the value of this area function at $x = 1$ is zero, as is expected. Also that its derivative is again the given function $f(x) = x^2$ for $x > 1$.

Let us display in the form of a table the area functions that we have calculated so far and their derivatives.

| Given function $f(x)$ | First Ordinate | Area function $A(x)$ (from first ordinate to $x$) | Derivative $A'(x)$ of the area function |
|---|---|---|---|
| $3x+1$ | 2 | $\frac{3x^2}{2}+x-8$ | $3x+1$ |
| $3x+1$ | 3 | $\frac{3x^2}{2}+x-\frac{33}{2}$ | $3x+1$ |
| $-2x+3$ | 1 | $-x^2+3x-2$ | $-2x+3$ |
| $-2x+3$ | 0 | $-x^2+3x$ | $-2x+3$ |
| $x^2$ | 0 | $\frac{x^3}{3}$ | $x^2$ |
| $x^2$ | 1 | $\frac{x^3-1}{3}$ | $x^2$ |

**Do you notice anything in this table ?** We see that its first column **and last column** are identical. **Would this be so if we had chosen some other functions in the first column ?** The answer to this question was given in the 17th century by the English **mathematician Isaac Barrow (1630-1677 A.D.), Isaac Newton (1642-1727 A.D.)** who, **incidentally, was** once a student of Barrow and the **German mathematician Gottfried Wilhelm Leibniz (1646-1716 A.D.)**. This answer is indeed one of the momentous achievements in Mathematics. We shall take it up in the next section.

## EXERCISE 29.1

Find the area function **for each** of the following functions from the **indicated** first ordinate. Verify that the **derivative of the area** function is the function $f(x)$.

1. $f(x)=2x-5,\ x=3$
2. $f(x)=2x-5,\ x=5$
3. $f(x)=-x+6,\ x=-3$
4. $f(x)=-x+6,\ x=2$

Find the area function for each of the following functions from the indicated first ordinate using Archimedes method. Verify that the derivative of the area function is the function $f(x)$.

5. $f(x)=x^2, \quad x=1$

6. $f(x)=x^2+3, \ x=0$

*7. $f(x)=x^2+x, \ x=0$

## 29.3 Integral as Area : Non-negative Functions

The method of Archimedes, though elegant, is rather cumbersome. Fortunately, Calculus provides us with other methods of finding areas bounded by more general curves than the Archimedes method.

We shall study these methods now.

Let $y=f(x)$ be a continuous function which takes only non-negative values i.e., the graph of $f(x)$ lies above the $x$–axis except possibly at some points where it may meet the $x$–axis. Let us consider the ordinates at $x=a$ and at some other point $x \geqslant a$. **(See Fig. 29.5)**

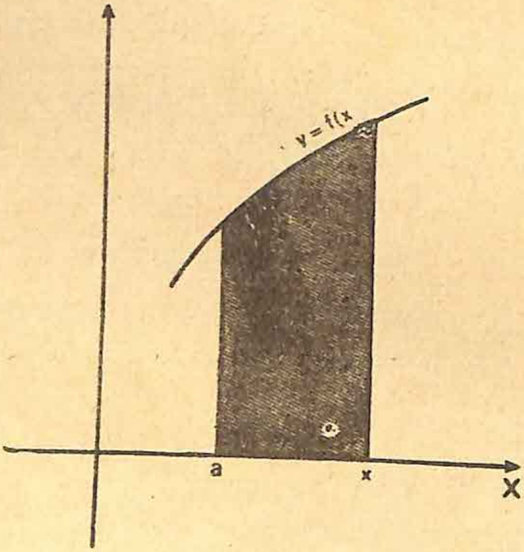

Fig. 29.5

*It is intuitively clear that the region bounded by the graph of $y = f(x)$, above the $x$-axis, between the ordinates at $x = a$ and at some other point $x$ has some area.* **How shall we compute this area ?** As remarked already the grid method has its drawbacks and Archimedes' method is too difficult, We, therefore, introduce some notations to be able to study the methods, provided to us by Calculus.

Since the area function depends on $x$ as well as the form of $f(x)$, we introduce a notation which takes into account both these facts. Following Leibniz, we write

$$A(x) = \int_a^x f(x)\,dx \tag{1}$$

and read the R.H.S. of (1) as **the integral from a to x of f(x) with respect to x** or simply **the integral of f(x) from a to x.**

Since in the function-notation, it does not matter what variable we use to express the form of the function, we can just as well write (1) as

$$A(x) = \int_a^x f(x)\,dx = \int_a^x f(t)\,dt = \int_a^x f(u)\,du$$

The symbol* $\int$ **is called the integral sign.** In this notation, **f (x) is called the integrand,** a is called the **lower limit of integration** and **x** is called the **upper limit of integration.** The process of finding the areas is called **integration.** Interval from a to $x$ is called the **interval or domain of integration.**

With this notation, we can write our area functions of Examples 1 and 2 as

$$\int_2^x (3x+1)\,dx = \frac{3}{2}x^2 + x - 8$$

$$\int_3^x (3x+1)\,dx = \frac{3}{2}x^2 + x - \frac{33}{2}$$

---

*The symbol is an elongated S, the first letter of the word 'Sum', since Archimedes' method involves limit of sums of areas of rectangles.

$$\int_1^x (-2x+3)\,dx = -x^2+3x-2$$

$$\int_0^x (-2x+3)\,dx = -x^2+3x$$

$$\int_0^x x^2\,dx = \frac{x^3}{3}$$

$$\int_1^x x^2\,dx = \frac{x^3-1}{3}$$

We also can write, for instance,

$$\frac{d}{dx}\left[\int_2^x (3x+1)\;dx\right] = 3x+1,$$

$$\frac{d}{dx}\left[\int_1^x (-2x+3)\;dx\right] = -2x+3,$$

$$\frac{d}{dx}\left[\int_0^x x^2\,dx\right] = x^2$$

The question which we raised at the end of Section 29.2 can now be restated terms of our new notation as follows :

**Is** $$\frac{d}{dx}\left[\int_a^x f(x)\,dx\right] = f(x)\ ?$$

We state the answer in the form of a theorem.

**Theorem 31 :** Let $f(t)$ be a non-negative, continuous function of $t$ for $a \leqslant t \leqslant b$. Let

$$A(x) = \int_{a}^{x} f(t)\,dt$$

Then $A'(x) = f(x)$ for all $x$ in $a \leqslant x \leqslant b$.

*To begin with we assume* that $f(x)$ is increasing in a small interval J around x.* [See Figs. 29.6 (i) and (ii) ]

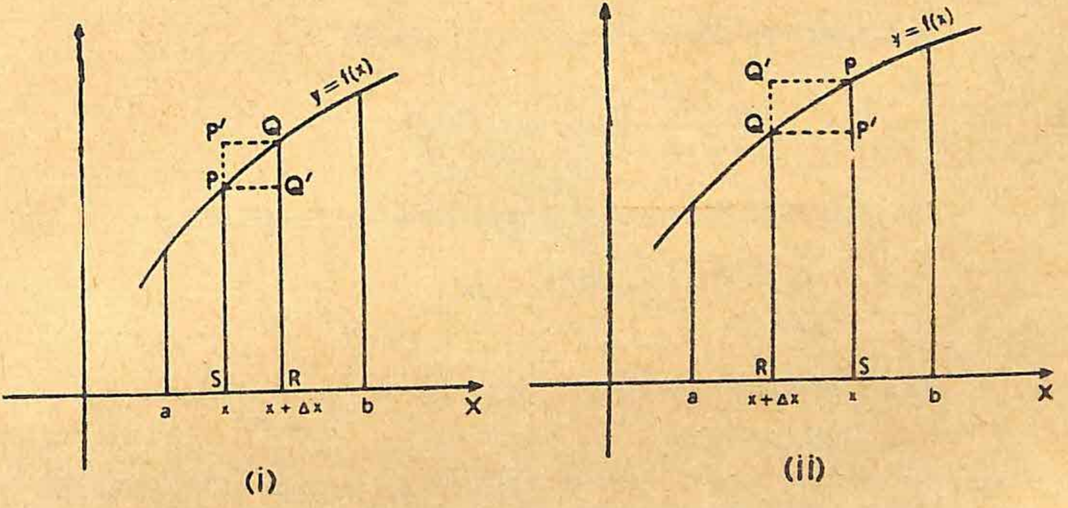

Fig. 29.6

We recall that $A(x)$ is the area under the function $f$ from a to $x$. Now how shall we calculate $A'(x)$ ?

$$A'(x) = \lim_{\Delta x \to 0} \frac{A(x+\Delta x) - A(x)}{\Delta x} \tag{1}$$

First we start with right-hand limit. Since we have to take limits, we assume $\Delta x$ (positive) to be small enough for the interval from $x$ to $x+\Delta x$ in J.

---

**The proof of the case when $f(x)$ is decreasing in a small interval around $x$ is left to the reader as an exercise.*

Now what is $A(x+\Delta x)$ ? It is the area under $f$ from a to $x+\Delta x$. Thus, for $\Delta x>0$, $A(x+\Delta x)-A(x)$ is the area of the region PQRS in Fig. 29.6 (i).

It is clear from Fig. 29.6 (i) that

$$ar\,(PQ'RS) < ar\,(PQRS) < ar\,(P'QRS)$$

i.e.,
$$f(x)\,\Delta x < A\,(x+\Delta x) - A\,(x) < f\,(x+\Delta x)\,\Delta x \qquad (2)$$

But, $\Delta x > 0$, thus we can rewrite (2) as

$$f\,(x) < \frac{A\,(x+\Delta x) - A(x)}{\Delta x} < f\,(x+\Delta x) \qquad (3)$$

By the Sandwich Theorem*, therefore, (3) gives us

$$\lim_{\Delta x \to 0+} \frac{A\,(x+\Delta x) - A(x)}{\Delta x} = f\,(x) \qquad (4)$$

The reader is advised to reason similarly from Fig. 29.6 (ii) and get

$$\lim_{\Delta x \to 0-} \frac{A\,(x+\Delta x) - A(x)}{\Delta x} = f\,(x) \qquad (5)$$

(4) and (5) together give us that

$$A'\,(x) = f\,(x)$$

which proves** the theorem.

[The reader is advised to furnish a proof of the theorem when $f$ is increasing on one side of $x$ and decreasing on the other in a small interval around $x$. Can you tell when will this happen ?]

This result is very powerful indeed, namely, that if $f\,(x)$ is a non-negative continuous function, then the derivative of its area function is $f\,(x)$. In the next section, we see

---

*Since f is a continuous function, it is clear that

$$\lim_{\Delta x \to 0+} f(x+\Delta x) = \lim_{\Delta x \to 0-} f(x+\Delta x) = f(x)$$

**In fact, the theorem can be proved without these restrictions on $f$. However we shall not take up that proof in this book.

that this result is true when the non-negativity of the function $f(x)$ is dropped, provided of course, we first define what we mean by $\int_a^x f(t)\,dt$ in that case.

## 29.4 Integral of a Function which may take negative values

Before we extend our definition of integral to include functions which may take negative values, let us introduce the concept of **signed area**. So far we have found areas of regions lying under a curve $y = f(x)$ and above the $x$-axis. **What if f is negative and we are concerned with the region which lies above the curve and below the x-axis?** (See Fig. 29.7)

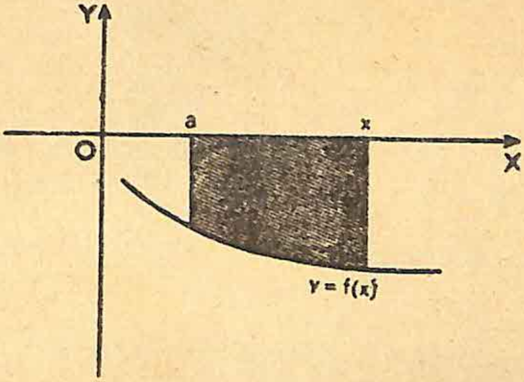

Fig. 29.7

Let the area of the region be A. To give it meaning as area under the curve and above the x-axis, we say that it has signed area —A. We write it as

$$\int_a^x f(x)\,dx = -A$$

and call it also as the integral from a to $x$.

Now we consider a function $f$ which may take positive as well as negative values. A graph of such a function is given in Fig. 29.8.

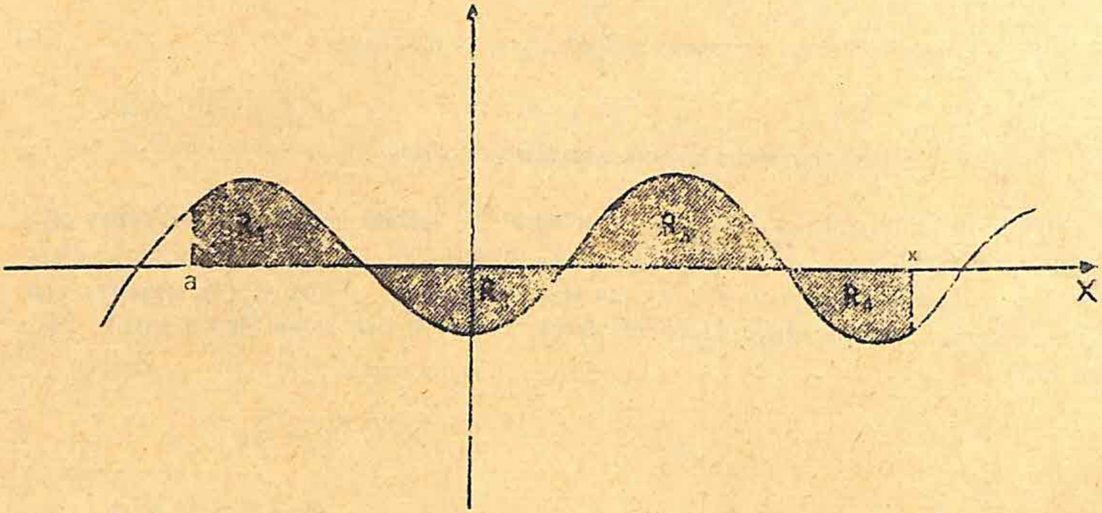

Fig. 29.8

The signed area under the curve $y = f(x)$ and above the x-axis is the area of the region $R_1$ − area of the region $R_2$ + area of the region $R_3$ − area of the region $R_4$.

We now extend the definition of integral to include such functions and write

$$\int_a^x f(x)\,dx = [ar\,(R_1) + ar\,(R_3)] - [ar\,(R_2) + ar\,(R_4)]$$

We consider an example.

**Example :** Compute

$$\int_{-4}^{x} (2x+1)\,dx$$

**Solution :** Recall that the graph of $y=2x+1$ is a line. It is clear that two cases arise according as $-4 \leqslant x \leqslant -\frac{1}{2}$ or $x > -\frac{1}{2}$.

**Case I :** $-4 \leqslant x \leqslant -\frac{1}{2}$

We see that the region lies below the x-axis. (See Fig. 29.9)

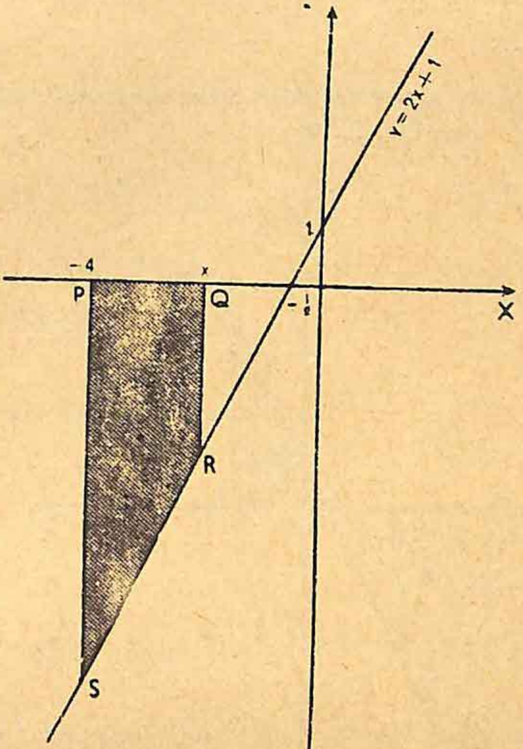

Fig. 29.9

We calculate $PS$, $QR$ and $PQ$ we have

$$PS = |f(-4)| = 7,\ QR = |2x+1| = -(2x+1)$$

$$PQ = |-4-x| = x+4$$

Thus,
$$ar\,(PQRS) = \frac{1}{2}\,[\,(PS+QR)\,PQ\,]$$

$$= \frac{1}{2}\,[\,(7-(2x+1))\,(x+4)\,]$$

$$= -x^2-x+12$$

Hence

$$\int_{-4}^{x} (2x+1)\,dx = -(-x^2-x+12) = x^2+x-12$$

when $-4 \leqslant x \leqslant -\frac{1}{2}$

**Case II :** $x > -\frac{1}{2}$

We see that the required area consists of areas of two regions, one lying below the x-axis and the other above. (See Fig. 29.10)

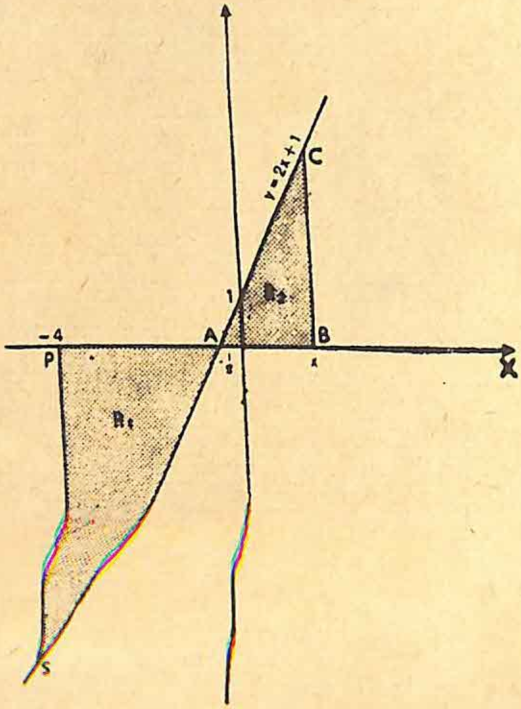

Fig. 29.10

Now, $ar\,(\text{APS}) = \frac{1}{2}\,(\text{PS})\,(\text{AP}) = \frac{1}{2}\,(7)\left(\frac{7}{2}\right) = \frac{49}{4}$

Thus, $ar\,(R_1) = \frac{49}{4}$

And, $ar\,(\text{ABC}) = \frac{1}{2}\,(\text{AB})\,(\text{BC}) = \frac{1}{2}\left(x + \frac{1}{2}\right)(2x+1)$

$$= x^2 + x + \frac{1}{4}$$

Thus, $ar\,(R_2) = x^2 + x + \frac{1}{4}$

We now have

$$\int_{-4}^{x} (2x+1)\,dx = -\,ar\,(R_1) + ar\,(R_2)$$

$$= -\frac{49}{4} + x^2 + x + \frac{1}{4}$$

$$= x^2+x-12$$

We see that in both the cases,

$$A\,(x) = \int_{-4}^{x} (2x+1)\,dx = x^2 + x - 12$$

We observe that $\frac{d}{dx}\,(x^2+x-12) = 2x+1 = f\,(x)$

i.e., $$\frac{d}{dx}\left[\int_{-4}^{x} (2x+1)\;dx\right] = 2x+1$$

This suggests that Theorem 31 is true for functions that may take negative values as well. In fact it is. We State this fact as a theorem.

**Theorem 32 : (First Fundamental Theorem of Integral Calculus)**

**Let $f\,(t)$ be a continuous function of $t$ for $a \leqslant t \leqslant b$.**

**Further, let $A\,(x) = \int_{a}^{x} f\,(t)\,dt$. Then $A'\,(x) = f\,(x)$ for all $x$ in $a \leqslant x \leqslant b$.**

## EXERCISE 29.2

Evaluate $\int_{a}^{x} f\,(x)\,dx$ for each of the following for the indicated value of $a$.

Verify that $\frac{d}{dx}\left[\int_{a}^{x} f\,(x)\;dx\right] = f\,(x)$

1. $f(x) = 2x+1, \quad a = -6$
2. $f(x) = 2x-5, \quad a = 0$
3. $f(x) = -x+6, \ a = 2$
4. $f(x) = -3x+1, \ a = 2$
5. Let $f(t)$ be a decreasing non-negative continuous function of $t$ defined for $a \leqslant t \leqslant b$.

   Let $A(x) = \int_a^x f(t)\,dt$, for $a \leqslant x \leqslant b$. Prove that $A'(x) = f(x)$

6. Let $f(t) = 2-t$. Evaluate $\int_x^4 f(t)\,dt$. Verify

$$\frac{d}{dx}\left[\int_x^4 f(t)\,dt\right] = -f(x)$$

## 29.5 Anti-derivatives

We now learn how to use the **First Fundamental Theorem of Integral Calculus** to compute the area functions. First, we make a definition

**Definition 11 : A function F(x) is said to be an anti-derivative or a primitive function of f (x) if**

$$\frac{d}{dx} F(x) = f(x)$$

**Remark :** We speak of **an** anti-derivative (and not **the** anti-derivative since **several functions may be anti-derivative of the same function.** Can you think of some examples ?

**If we know one anti-derivative of a given function f (x), can we write several others ?** Consider, for instance, the following example.

**Example 1** : Since $\frac{d}{dx}\sin x = \cos x$, therefore, *sin x is an anti-derivative of cos x.*

It is clear that $\sin x + 1$, $\sin x - 15$, $\sin x + \frac{3}{2}$ are also anti-derivatives of $\cos x$.

**In general, we say sin x + C, where C is a real constant, is an anti-derivative of cos x.**

*Are there any other anti-derivatives of cos x, other than sin $x + C$?* The following theorem tells us that there are no others.

**Theorem 33 : Let f(x) and g (x) be differentiable functions on an interval a $\leqslant$ x $\leqslant$ b such that f′(x)=g′(x) for all x in this interval. Then there exists a real constant c such that f(x)=g(x)+c.**

We wish to prove that $f(x)-g(x)$ is a constant $c$.

Let us consider the function

$$\varphi(x) = f(x) - g(x)$$

We shall prove that if $x_1$ and $x_2$ are any two points in the interval a $\leqslant x \leqslant b$, then

$$\varphi(x_1) = \varphi(x_2)$$

Let us consider $x_1$ and $x_2$ in a $\leqslant x \leqslant b$ such that $x_1 < x_2$.

Since $f$ and $g$ are differentiable in the interval a $\leqslant x \leqslant b$, it is clear that $\varphi$ is differentiable in a $\leqslant x \leqslant b$. In particular, $\varphi$ is continuous in a $\leqslant x \leqslant b$.

We apply now **Lagrange's Mean Value Theorem** to the function $\varphi(x)$ in the interval $x_1 \leqslant x \leqslant x_2$. It tells us that there is at least one point, say $x_0$, in the interval a < x < b where the slope of the secant from $\Big(x_1, \varphi(x_1)\Big)$ to $\Big(x_2, \varphi(x_2)\Big)$ is equal to the slope of the tangent at $(x_0, \varphi(x_0))$. Thus we have,

$$\frac{\varphi(x_2)-\varphi(x_1)}{x_2-x_1} = \varphi'(x_0) \tag{1}$$

where $x_0$ is such that $x_1 \leqslant x_0 \leqslant x_2$.

But $\varphi'(x_0) = f'(x_0)-g'(x_0)$ (Why ?) (2)

Since $f'(x_0) = g'(x_0)$, by hypothesis, (2) gives us $\varphi'(x_0) = 0$.

From (1), therefore, we have

$$\frac{\varphi(x_2)-\varphi(x_1)}{x_2-x_1} = 0$$

whence, $\varphi(x_2)=\varphi(x_1)$;

which proves the theorem.

**Remark** : It follows, therefore, that whenever we know **one** anti-derivative F($x$ of some function $f(x)$, we know **all the others.** They are of the form $F(x)+C$, where C is some real constant. Thus, **any two anti-derivatives of a given function differ by a constant.**

This gives us a clue to **finding the area function through anti-derivatives.** Let us consider the following examples :

**Example 2 :** Compute the area function, A($x$), where

$$A(x) = \int_{1}^{x} x^3 dx$$

Can you guess an anti-derivative of $x^3$ ? We recall that

$$\frac{d}{dx}(x^4) = 4x^3 \text{ and hence,}$$

$$\frac{d}{dx}\left(\frac{x^4}{4}\right) = x^3$$

This tells us that $\frac{x^4}{4}$ is an anti-derivative of $x^3$.

On the other hand, by the **First Fundamental Theorem of Integral Calculus,** A($x$) is also an anti-derivative of $x^3$. Thus by Theorem 33 above,

$$A(x) = \frac{x^4}{4} + c \qquad (1$$

where $c$ is some suitable constant.

How shall we evaluate $c$ ? We recall that A(1)=0. Substituting $x=1$ in (1) we have

$$A(1) = \frac{1}{4} + c$$

Thus, A(1) = 0 implies that $c = -\frac{1}{4}$ and that

$$A(x) = \frac{x^4 - 1}{4}$$

**Example 3** : Compute the area function, $A(x)$, where

$$A(x)=\int_{2}^{x} x^3dx$$

**Solution** : From Example 2, above, we have

$$A(x) = \frac{x^4}{4} + c$$

where c is to be suitably chosen so that $A(2)=0$.

Now, $A(2) = \frac{(2)^4}{4} + c$

Thus, $A(2) = 0$ implies that $c = -4$.

Thus, $A(x)=\frac{x^4-16}{4}$

We see that we get **different area functions for different values of the lower limit of integration.** They are all of the form

$$A(x) = \frac{x^4}{4} + c$$

## 29.6 The Definite Integral

Let us refer to Example 3, Section 29.5. We had computed

$$A(x) =\int_{2}^{x} x^3dx = \frac{x^4}{4} - 4 \qquad (1)$$

In (1), if we substitute a specific value of $x$, say $x = 4$, we get

$$A(4)=\int_{2}^{4} x^3dx = \frac{(4)^4}{4} - 4 = 60$$

This specific value of the area function is called the **definite integral of $x^3$ from 2 to 4.**

More generally, **if a and b are some (fixed) numbers, such that a < b, and f is a continuous function defined on an interval $a \leqslant x \leqslant b$, then the definite integral of f from a to b is**

$$\int_a^b f(x)dx \quad \text{or} \quad \int_a^b f(t)dt$$

In other words, **it is the area under the graph of $y=f(x)$ between the ordinate at $x=a$ and at $x=b$.**

**Remark:** The definite integral $\int_a^b f(x)dx$ is simply **the value of the area function $A(x)$,**

**where,**

$$A(x) = \int_a^x f(x)dx$$

**at the point $x=b$.**

We prove below an important theorem which enables us to evaluate definite integrals by making use of the anti-derivatives. The theorem is called the **Second Fundamental Theorem of Integral Calculus.**

**Theorem 34 : Let f be a continuous function defined on the interval $a \leqslant x \leqslant b$ and let F be an anti-derivative of f. Then**

$$\int_a^b f(x)dx = F(b)-F(a)$$

**Proof :** Since we are dealing with area, which is a specific value of the area function, we let

$$A(x) = \int_a^x f(x)dx$$

By the **First Fundamental Theorem of Integral Calculus**, A is an anti-derivative of $f$. But F is also an anti-derivative of $f$, by hypothesis. Thus, by Theorem 33

$$A(x) = F(x)+c \tag{1}$$

where $c$ is a real constant.

Now $A(a)$ must be zero. Thus, from (1), we have

$$F(a)+c=0$$

whence, $\quad c= -F(a)$

(1), therefore, becomes

$$A(x)=F(x) -F(a) \tag{2}$$

In (2), let us substitute $x=b$. We get

$$A(b)=F(b)-F(a)$$

But $A(b)=\int_a^b f(x)dx$

We thus have

$$\int_a^b f(x)dx=F(b)-F(a)$$

which proves the desired result.

**Remark 1 :** In words, the theorem tells us that

$$\int_a^b f(x)dx = \text{(Value of an anti-derivative at } b\text{, the upper limit)} - \text{(Value of the same anti-derivative at } a\text{, the lower limit)}$$

**Remark 2 :** We often write $F(b)-F(a)$ as $F(x)\Big]_a^b$

**Remark 3 :** It should be clear that, no matter which anti-derivative we take as F the value of the definite integral comes out to be the same.

We now consider some examples.

**Example 1 :** Evaluate

$$\int_2^3 (t^2+1)dt$$

**Solution :** An anti-derivative of $t^2+1$ is $\frac{t^3}{3}+t$

Thus by the Second Fundamental Theorem

$$\int_2^3 (t^2+1)\,dt = \left(\frac{t^3}{3}+t\right)\Big]_2^3 = \left(\frac{3^3}{3}+3\right) - \left(\frac{2^3}{3}+2\right)$$

$$= 12 - \frac{14}{3} = \frac{22}{3}$$

Hence, $\int_2^3 (t^2+1)\,dt = \frac{22}{3}$

**Example 2 :** Find $\int_0^4 t^{\frac{3}{2}}\,dt$

**Solution :** An anti-derivative of $t^{\frac{3}{2}}$ is $\frac{t^{\frac{5}{2}}}{\frac{5}{2}} = \frac{2}{5}t^{\frac{5}{2}}$

Thus, $\int_0^4 t^{\frac{3}{2}}\,dt = \frac{2}{5}\; t^{\frac{5}{2}}\Big]_0^4$

$$= \frac{2}{5}\left(4^{\frac{5}{2}} - 0\right)$$

$$= \frac{2}{5}\left(2^5 - 0\right)$$

$$= \frac{64}{5}$$

Hence, $\int_{0}^{4} t^{\frac{3}{2}}\, dt = \frac{64}{5}$

**Example 3** : Evaluate $\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \cos t\, dt$

**Solution** : An anti-derivative of $\cos t$ is $\sin t$.

$$\text{Therefore,}\quad \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \cos t\, dt = \sin t \Big]_{\frac{-\pi}{2}}^{\frac{\pi}{2}}$$

$$= \sin\frac{\pi}{2} - \sin\left(-\frac{\pi}{2}\right)$$

$$= \sin\frac{\pi}{2} + \sin\frac{\pi}{2} = 1+1=2$$

Hence, $\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \cos t\, dt = 2$

**Remark:** The result of Example 3 tells us that the area under the graph of $y=\cos x$ between $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{2}$ is 2. (See Fig. 29.11)

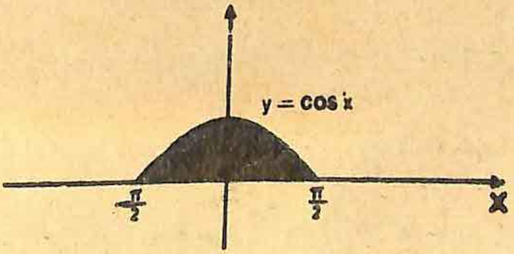

Fig. 29.11

## 29.7 Properties of the Definite Integral

We consider some important properties of the definite integral which we will find useful in evaluating them. As a motivation for proving these properties, the reader is advised to recall the properties of **the area function** from earlier classes.

**Property 1 : If $\mathbf{f(x) = k}$, where k is a real constant then**

$$\int_{a}^{b} \mathbf{f(x)\,dx = k(b-a)}$$

Since $F(x)=kx$ is an anti-derivative of $f(x)=k$, we have

$$\int_{a}^{b} f(x)dx = kx\Big]_{a}^{b} = kb-ka = k(b-a)$$

**Property II : If $\mathbf{a < c < b}$ and if f is a continuous function defined on $\mathbf{a \leqslant x \leqslant b}$, then**

$$\int_{a}^{b} \mathbf{f(x)\,dx} = \int_{a}^{c} \mathbf{f(x)\,dx} + \int_{c}^{b} \mathbf{f(x)\,dx}$$

Let F be an anti-derivative of $f$. then, by the Second Fundamental Theorem,

$$\int_{a}^{b} f(x)\,dx = F(b) - F(a) \qquad \text{(i)}$$

$$\int_{a}^{c} f(x)\,dx = F(c) - F(a) \qquad \text{(ii)}$$

$$\int_{c}^{b} f(x)\,dx = F(b) - F(c) \qquad \text{(iii)}$$

Adding (ii) and (iii), we have

$$\int_{a}^{c} f(x)\,dx + \int_{c}^{b} f(x)\,dx = F(b) - F(a) = F(x)\Big]_{a}^{b} = \int_{a}^{b} f(x)dx$$

Intuitively, the property tells us that the area under $y=f(x)$ between $x=a$ and $x=b$ is the sum of the areas from $x=a$ to $x=c$ and from $x=c$ to $x=b$, where $c$ is some point in $a < x < b$. (See Fig. 29.12)

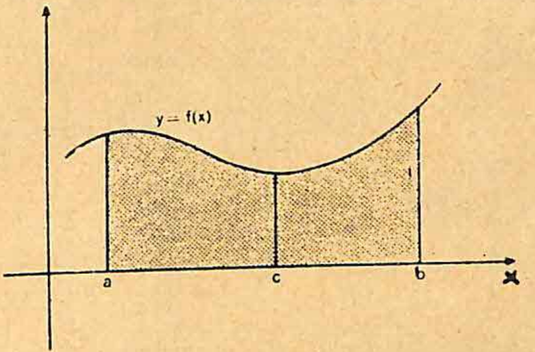

Fig. 29.12

This property is called the **Additivity Property of the Definite Integral of a function.**

**Property III : Let f and g be continuous on a $\leqslant$ x $\leqslant$ b.**

**Further let k be a constant. Then**

$$\text{(i)}\quad \int_a^b [f(x)+g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

$$\text{(ii)}\quad \int_a^b k\, f(x)dx = k\int_a^b f(x)dx$$

Let F and G be anti-derivatives of $f$ and $g$, respectively. Recall that

$$\frac{d}{dx}[F(x) + G(x)] = \frac{d}{dx}F(x) + \frac{d}{dx}G(x) = f(x)+g(x)$$

Thus, F + G is an anti-derivative of $f+g$

Hence,

$$\int_a^b [f(x)+g(x)]\, dx = [F(x) + G(x)]\Big]_a^b$$

$$= [\ F(b)+G(b)\ ] - [\ F(a)+G(a)\ ]$$

$$= [\ F(b)-F(a)\ ] + [\ G(b)-G(a)\ ]$$

$$= F(x)\Big]_a^b + G(x)\Big]_a^b$$

$$= \int_a^b f(x)dx + \int_a^b g(x)\,dx$$

Thus,

$$\int_a^b [f(x)+g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

This proves (i). It tells us that **the definite integral of the sum of two functions is the sum of their separate definite integrals.**

For (ii), we recall that

$$\frac{d}{dx}\,[k\ F(x)] = k\frac{d}{dx}\ F(x) = k\,f(x)$$

Thus, $\int_a^b k\,f(x)\,dx = k\ F(x)\Big]_a^b = k\ F(b) - k\ F(a) = k\ [F(b) - F(a)\,]$

i. e. $\int_a^b k\,f(x)\,dx = k\int_a^b f(x)\,dx$

which proves (ii). It tells us that **the definite integral of a constant multiple of a function is that constant multiple of the definite integral of the function.**

By repeated application of (i) and (ii) of Property III, it is straightforward to prove the following :

**Theorem 35 : If $f_1, f_2, \ldots, f_n$ are continuous on $a \leqslant x \leqslant b$ and if $k_1, k_2, \ldots, k_n$ are constants, then**

$$\int_a^b [k_1 f_1(x) + k_2 f_2(x) + \ldots + k_n f_n(x)]\, dx$$

$$= k_1 \int_a^b f_1(x)\, dx + k_2 \int_a^b f_2(x)\, dx + \ldots + k_n \int_a^b f_n(x)\, dx$$

We call this the **Linearity Property of the Definite Integral** of Several Functions.

Finally, we make an important definition.

**Definition 12 : If $a < b$**

$$\int_b^a f(x)\, dx = -\int_a^b f(x)\, dx$$

This definition tells us, for instance, that

$$\int_3^1 t^2\, dt = -\int_1^3 t^2\, dt = -\,8\frac{2}{3}$$

**Remark 1 :** We can now extend the **Additivity Property** for $c$ lying outside the interval $a < x < b$.

**Remark 2 :** We leave it for the reader to verify that the Second Fundamental Theorem, Properties I, II, III and Theorem 35 continue to hold for the definite integral

$$\int_b^a f(x)\, dx, \; a < b$$

We only need some method(s) of finding anti-derivatives to be able to evaluate definite integrals. In the next section, we develop some basic anti-derivatives and consider some examples.

## 29.8 Finding Anti-Derivatives. Indefinite Integrals

Anti-derivatives are also called **Indefinite Integrals** or, simply **Integrals. We denote indefinite integral of f(x) by the symbol**

$$\int f(x)\, dx$$

and read it as *the integral of $f(x)$ with respect to $x$.* The process of finding an anti-derivative is called **Integration.**

We should remark that since a function $f$ has several anti-derivatives, $\int f(x)\, dx$ is rather ambiguous in meaning, in the sense that it does not tell us which anti-derivative is intended. **It is for this reason that, if F is an anti-derivative of f, we write**

$$\int \mathbf{f(x)\, dx = F(x) + c}$$

**where c is a real constant** and is called the **Constant of Integration.** For different values of c, we get different anti-derivatives of f, and hence **all** anti-derivatives of $f$.

We consider some examples.

**Example 1 :** Find $\int \sin x\, dx$

**Solution :** Since $\frac{d}{dx}(-\cos x) = \sin x$, $(-\cos x)$ is an anti-derivative of $\sin x$. Hence we write

$$\int \sin x\, dx = -\cos x + c, \quad c \text{ a constant}$$

**Example 2 :** Integrate $\sec^2 x + 2x$

**Solution :** We recall that $\frac{d}{dx}(\tan x) = \sec^2 x$ and $\frac{d}{dx}(x^2) = 2x$

Also that

$$\frac{d}{dx}(\tan x + x^2) = \sec^2 x + 2x$$

Hence,

$$\int (\sec^2 x+2x)\,dx = \tan x+x^2+c$$

where $c$ is the constant of integration.

**Remark** : Let us now evaluate $\int \sec^2 x\,dx+2\int x\,dx$

**We have**

$$\int \sec^2 x\,dx +2\int x\,dx = (\tan x+c_1) + (2.\,\frac{x^2}{2}+c_2)$$

$$= \tan x+x^2+c_1+c_2$$

$$\tan x+x^2+c, \text{ where}$$

$c = c_1+c_2$. **It is customary to combine all constants and write them as one constant of integration.**

We observe that

$$\int (\sec^2 x + 2x)\,dx = \int \sec^2 x\,dx +2\int x\,dx$$

We state below **two important properties of indefinite integrals,** which we find of great use.

**If f and g are continuous and k is a constant then**

(i) $\int [\,f(x)+g(x)\,]\,dx = \int f(x)\,dx + \int g(x)\,dx$

(ii) $\int k\,f(x)\,dx = k\int f(x)\,dx$

We now give a table of integrals (anti-derivatives) of some functions. This table is easy to construct from our earlier knowledge of Calculus.

$$\int x^n dx = \frac{x^{n+1}}{n+1}+c,\ n\neq -1$$

$$\int \cos x\,dx = \sin x+c$$

$$\int \sin x\,dx = -\cos x + c$$

$$\int \sec^2 x\,dx = \tan x + c$$

$$\int \text{cosec}^2 x\ dx = -\cot x + c$$

$$\int \sec x \tan x\ dx = \sec x + c$$

$$\int \text{cosec}\ x \cot x\ dx = -\text{cosec}\ x + c$$

We consider some examples where we use the properties and results from the above table.

**Example 3** : Find $\int (2 \sin x + x^2)\ dx$

**Solution** : $\int (2 \sin x + x^2) dx = 2\int \sin x\ dx + \int x^2 dx$

$$= 2(-\cos x) + \frac{1}{3}x^3 + c$$

$$= -2 \cos x + \frac{x^3}{3} + c$$

**Thus,**

$$\int (2 \sin x + x^2)\ dx = -2 \cos x + \frac{x^3}{3} + c$$

**Example 4** : Integrate $\sqrt{z^3} - 3 \sec^2 z$

**Solution** : $\int (\sqrt{z^3} - 3 \sec^2 z)\ dz = \int \sqrt{z^3}\ dz - 3\int \sec^2 z\ dz$

$$= \int z^{\frac{3}{2}}\ dz - 3 \int \sec^2 z\ dz$$

$$= \frac{z^{\frac{3}{2}+1}}{\frac{3}{2}+1} - 3 \tan z + c$$

$$= \frac{2}{5}\ z^{\frac{5}{2}} - 3 \tan z + c$$

Thus, $$\int ( \sqrt{z^3} - 3 \sec^2 z ) \, dz = \frac{2}{5} z^{\frac{5}{2}} - 3 \tan z + c$$

**Example 5 : Find all functions f whose second derivative is zero for all x in the domain of f.**

**Solution** : We desire all those functions $f(x)$ such that

$$f''(x) = 0$$

for all $x$ in the domain of $f$.

[In Leibniz's notation, this may be written as $\frac{d^2y}{dx^2} = 0$, where $y = f(x)$. Such an equation involving derivatives (differential coefficients) is called a **differential equation.]** Recall that

$$\frac{d^2y}{dx^2} = \frac{d}{dx} \left( \frac{dy}{dx} \right)$$

Thus, if $\frac{d^2y}{dx^2} = 0$, we must have

$$\frac{dy}{dx} = a \tag{1}$$

where $a$ is some constant. Again taking anti-derivatives in (1) and noting that $y$ is an anti-derivative of the L. H. S., we get

$$y = \int a \, dx$$

$$y = \alpha x + \beta$$

where $\beta$ is the constant of integration.

We observe that $y$ is a linear function of $x$. We have shown that, if for a function $y = f(x)$,

$$\frac{d^2y}{dx^2} = 0,$$

then $y$ must be a linear function of $x$.

Conversely, it is easy to see that if $y$ is a linear function of $x$ then

$$\frac{d^2y}{dx^2} = 0$$

**Thus, only linear functions of x and no other functions, have their second derivatives zero for all values of x.**

**Remark 1** : In the above example, we say that $y = \alpha x + \beta$ **satisfies** the differential equation $\frac{d^2y}{dx^2} = 0$. **A function satisfying a given differential equation is called a solution of the differential equation.**

**Remark 2 :** We note that $y = \alpha x + \beta$ represents **all** solutions of the differential equation $\frac{d^2y}{dx^2} = 0$.

**Remark 3 :** *We observe that there are two arbitrary constants in the solution of the differential equation* $\frac{d^2y}{dx^2} = 0$.

**Example 6 :** Find $\int_0^{\frac{\pi}{2}} (x + \cos x)\, dx$

**Solution** : We have

$$\int_0^{\frac{\pi}{2}} (x + \cos x)\, dx = \int_0^{\frac{\pi}{2}} x\, dx + \int_0^{\frac{\pi}{2}} \cos x\, dx$$

$$= \left.\frac{x^2}{2}\right]_0^{\frac{\pi}{2}} + \left.\sin x\right]_0^{\frac{\pi}{2}}$$

$$= \frac{\left(\frac{\pi}{2}\right)^2}{2} - \frac{0^2}{2} + \left[\sin\left(\frac{\pi}{2}\right) - \sin(0)\right]$$

$$= \frac{\pi^2}{8} + 1$$

Thus, $$\int_0^{\frac{\pi}{2}} (x+\cos x)\, dx = \frac{\pi^2}{8}+1$$

## EXERCISE 29.3

Integrate the following functions :

1. $5x^2+6\cos x$

2. $\frac{10}{x^3} - 2\sec^2 x$

3. $6x + 5x^2 - 2x^3 + \frac{1}{x^2}$

4. $6\,\text{cosec}^2 x - \frac{1}{x^5}$

5. $2\sin x - \sec^2 x + x^{-\frac{3}{2}}$

6. $\tan^2 x - 3x^2$

7. $\frac{4}{x^2} - 2x^{\frac{1}{2}} + \cot^2 x$

8. We have already studied motion of bodies moving with constant acceleration. The ancient Greeks believed the wrong notion that heavier the body, the farther it falls—a belief propagated by **Aristotle(384—322 B.C.)**, which remained unchallenged until the time of the **Italian genius Galilei Galileo (1564-1642 A.D.)**. Galileo made towering contributions to mathematics, physics, astronomy and claims priority over **Newton** and **Leibniz** as setting the pace for developments in Calculus. **Galileo** stated the **law of freely falling bodies** as follows :

**A body falling freely under gravity, starting from rest, travels a distance proportional to the square of the time t it has been falling, the constant of proportionality being the same for all bodies.**

*Assuming a constant acceleration g, due to gravity, prove Galileo's law.*

[**Hint** : Recall that $\frac{ds}{dt}$ is the **velocity**. The rate of change of velocity with respect to time is called the **acceleration**. Thus, acceleration $= \frac{d}{dt}\left(\frac{ds}{dt}\right) = \frac{d^2s}{dt^2}$ In our question, $\frac{d^2s}{dt^2} = g$ and $s=0$, when $t=0$. Also, $\frac{ds}{dt} = 0$ when $t=0$.]

9. (a) Show that the distance $s$, travelled in time $t$, by a body falling freely under gravity (due to a constant acceleration $g$) and starting with the initial velocity $u$ is

$$s = ut + \frac{1}{2}gt^2$$

[**Hint** : $\frac{ds}{dt} = u$, when $t=0$. Also, $s=0$ when $t=0$.]

(b) How long will it take for an object thrown vertically downwards from a height of 56.10 metres, with an initial velocity of 4m sec, to reach the ground ? Determine also the velocity with which it hits the ground (Take $g = 9.8$ $m$ $sec^2$).

10 . Find all functions $f$ whose third derivative is zero for all $x$ in the domain of $f$.

[Recall that $f''$ is called the **second derivative** of $f$. Derivative of $f''$ is called the **third derivative** of $f$. We denote it by $f'''$. In Leibniz's notation, this may be written as $\frac{d^3y}{dx^3} = 0$, where $y=f(x)$.]

[We will see that the **solution of $f'''(x) = 0$ contains three arbitrary constants.**]

**Calculate the signed area under the graph of the following functions between the indicated ordinates and above the $x$-axis.**

**11.** $f(x) = 3x-7$ ; $x=0$, $x=3$

**12.** $f(t) = 3-2t$ ; $t=-1$, $t=3$

**13.** $\phi(z) = z^3+5z^2-2z-\frac{1}{z^2}$ ; $z=1$, $z=2$

**14.** $f(x) = \sin^2\frac{x}{2}$ ; $x=0$, $x=\frac{\pi}{2}$

Evaluate $\int_a^b f(x)\,dx$ for each of the following :

15. $f(x) = \sqrt{2x} - \frac{1}{\sqrt{2x}},\ a=1,\ b=2$

16. $f(x) = x^{\frac{2}{3}} + \cos x,\ a=0,\ b = \frac{\pi}{3}$

17. $f(x) = \frac{x^4+2x^2-1}{\sqrt{x}},\ a = 1,\ b=4$

Evaluate each of the following :

18. $\int\left(\sqrt{x} - \cos^2\frac{x}{2}\right) dx$

19. $\int\left(x^{\frac{3}{2}} - \sin\frac{x}{2}\ \cos\frac{x}{2} + 1\right) dx$

20. $\int (3 \operatorname{cosec}^2 x - 5x + \sin x)\, dx$

21. $\int (\sec^2 x + \cos x)\, dx$

22. $\int\left(\sin x + \cos x + x^{\frac{5}{2}}\right) dx$

Evaluate each of the following :

23. $\int_2^t \frac{x^2-2x+1}{x^4}\, dx$

24. $\int_0^{\frac{\pi}{2}} (5 \sin x + 2 \cos x)\, dx$

25. $\int_0^{\frac{\pi}{4}} (2 \sec^2 x + x^3 + 2)\, dx$

**26.** $\int_{0}^{\pi} \left( \sin^2 \frac{x}{2} - \cos^2 \frac{x}{2} \right) dx$

Integrate the following functions :

**27.** $x^{\frac{4}{3}} - 2 \sin x + 2$

**28.** $\frac{1}{x^2} + \sec^2 x + 5x$

**29.** $\cos x + \operatorname{cosec}^2 x - x^8 + 1$

**30.** $x^6 - \cos x + \sec^2 x + \sqrt{x}$

## 29.9 Key Concepts

Area function

Area under $y=f(x)$,
above the x-axis and
between two given ordinates

Squaring the Parabola
(Archimedes' Theorem)

Integral as area

Integral

Integrand

Limits of integration
—Lower and upper limits

Interval or domain of integration

Signed area

First Fundamental Theorem of
Integral Calculus

Anti-derivative or primitive functions

The definite integral

Second Fundamental Theorem of Integral
Calculus

Properties of the definite integral
—Additivity
—Linearity

Indefinite integral

Constant of integration

Differential equations

Second derivative

Third derivative

Velocity

Acceleration

## 29.10 Suggestions for Further Reading

*Any standard text in Calculus will have materials on integration. We recommend the following :*

[3] of Section 8.6., [1] of Section 18.7, [1] of Section 20.4, [1] of Section 21.9 and [1] of Section 24.5.

*The reader is also referred to*

[1] P.R. Masani, etc. **Elementary Calculus.** Second Revised Edition. Popular Prakashan, Bombay, (India). 1975.

# UNIT XXX

## TECHNIQUES OF INTEGRATION

*So far we are able to integrate a few, rather simple functions. We now learn some techniques of integration to be able to integrate somewhat more complicated functions which we encounter in mathematics and its applications.*

### 30.1 Introduction

The table of integrals of Section 29.8 helps us to integrate some very simple functions, viz., those whose anti-derivatives can be readily guessed. For instance, an anti-derivative of $\sin x$ is $-\cos x$ and this helps us to find the area under the curve $y = \sin x$ between two ordinates.

However, let us refer to Question 3, Exercise 25.4. The reader was asked to draw graphs of $\sin x$ and $\sin 2x$ on the same axes. (See Fig. 30.1)

*What if now we are to say which graph has a larger area under it, say, from* $x=0$ *to* $x = \frac{\pi}{2}$ ? To determine this, we will need the **area functions**. Of course, we already know the area function of $\sin x$.

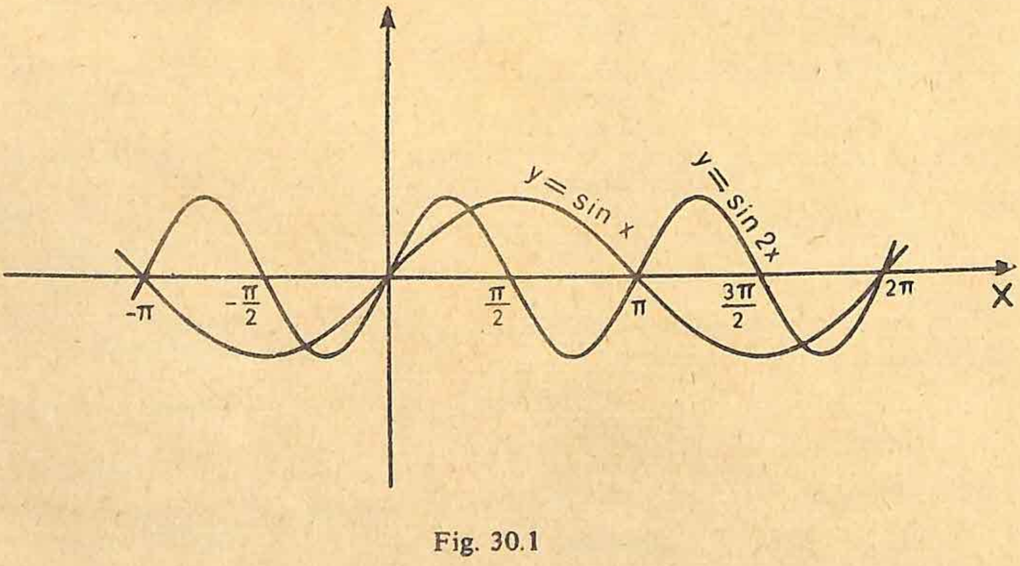

Fig. 30.1

**To find the area function of sin 2x, we need an anti-derivative of sin 2x.** How shall we proceed ?

A moment's reflection suggests that an anti-derivative of sin $2x$ is $-k\cos 2x$ where $k$ is a constant to be suitably chosen so that the derivative of $-k\cos 2x$ should be the original function sin $2x$.

$$\text{Now, } \frac{d}{dx}(-k\cos 2x) = 2k\sin 2x$$

$$\text{Thus, } 2k = 1 \text{ or, } k = \frac{1}{2}.$$

**Hence, an anti-derivative of sin 2x is $-\frac{1}{2}\cos 2x$.** If we let $A_1$ and $A_2$ denote respectively the two areas from 0 to $\frac{\pi}{2}$, we have

$$A_1 = \int_0^{\frac{\pi}{2}} \sin x\,dx = -\cos x\Big]_0^{\frac{\pi}{2}} = -\cos\frac{\pi}{2} - (-\cos 0) = 1$$

$$A_2 = \int_0^{\frac{\pi}{2}} \sin 2x\,dx = -\frac{1}{2}\cos 2x\Big]_0^{\frac{\pi}{2}} = -\frac{1}{2}\cos\pi - \left(-\frac{1}{2}\cos 0\right) = 1$$

**The two areas are thus equal.**

*How do the two areas compare*, say, *from* $x = \frac{\pi}{4}$ *to* $x = \frac{\pi}{3}$ ? The reader is advised to compute the two areas.

What do you think will be an anti-derivative of cos $2x$ ? It is easy to show **that an anti-derivative of cos 2x is $\frac{1}{2}$ sin 2x.**

We state the two anti-derivatives as formulae :

$$\int \sin 2x\,dx = -\frac{1}{2}\cos 2x + c$$

$$\int \cos 2x\,dx = \frac{1}{2}\sin 2x + c$$

We consider some more examples.

**Example 1 :** Find the area under the curve $y = \sin^2 x$ from 0 to $x$.

**Solution :** Can we express $\sin^2 x$ as a function or a sum of functions whose anti-derivatives we already know ?

We recall that

$$\sin^2 x = \frac{1}{2} - \frac{\cos 2x}{2}$$

Using Theorem 35, we have

$$\int_0^x \sin^2 x \, dx = \int_0^x \frac{1}{2} \, dx - \frac{1}{2} \int_0^x \cos 2x \, dx$$

$$= \frac{1}{2} x \Big]_0^x - \frac{1}{2} \frac{\sin 2x}{2} \Big]_0^x$$

$$= \frac{1}{2} x - \frac{1}{4} \sin 2x$$

Thus, $\displaystyle\int_0^x \sin^2 x \, dx = \frac{1}{2} x - \frac{1}{4} \sin 2x$

**Remark :** Suppose now that we are asked to find an integral of $\sin x^2$. By Theorem '5, $\sin x^2$ is differentiable and hence, continuous. Thus, by the **First Fundamental Theorem of Integral Calculus**, $\int \sin x^2 \, dx$ exists.

*Can $k \cos x^2$ be an anti-derivative of $\sin x^2$, with the constant $k$ suitably chosen ?* Let us see. Now,

$$\frac{d}{dx} (k \cos x^2) = -2x (k \sin x^2) \qquad (1)$$

**Thus $k \cos x^2$ can not be an anti-derivative of $\sin x^2$ for any value of k.** In fact, with all the techniques that we shall develop in this book, it will not be possible to express $\int \sin x^2 \, dx$ in terms of elementary functions known to us.

**Note :** In (1), if we let $k = -1$, we obtain the following interesting result :

$$\int 2x \sin x^2 \, dx = -\cos x^2 + c$$

[The reader can take the above as a clue to integrate functions, such as, $2x \cos x^2$, $2x \sec x^2$, etc. We, of course, would give later a more systematic method of integrating such functions.]

**Example 2 :** Find an integral of $f(x) = x \sin (x^2+2)$.

**Solution :** Clearly, $f(x)$ is differentiable and hence, continuous. By the **First Fundamental Theorem of Integral Calculus,** therefore, $\int x \sin (x^2+2) \, dx$ exists. However, it can not be readily guessed.

How shall we proceed ? We wish to express $f(x)$ as a derivative of some function. The fact that $(x^2+2)$ is the argument of sine suggests that we let $x^2+2=t$. $\sin (x^2+2)$ can then be written as $\sin t$.

Now, $\sin t = \frac{d}{dt} (-\cos t)$

And, $x = \frac{1}{2} \frac{d}{dx} (x^2+2) = \frac{1}{2} \frac{dt}{dx}$

Thus, $f(x) = \frac{1}{2} \frac{dt}{dx} \frac{d}{dt} (-\cos t) = \frac{d}{dt}\left(-\frac{1}{2} \cos t\right)\frac{dt}{dx}$

which, by the **Chain Rule,** is

$$= \frac{d}{dx}\left(-\frac{1}{2} \cos t\right) = \frac{d}{dx}\left[-\frac{1}{2} \cos (x^2+2)\right]$$

Thus, $\int x \sin (x^2+2) \, dx = -\frac{1}{2} \cos (x^2+2) + c$

**Example 3 :** Integrate $f(x) = 2x (x^2+2)^{\frac{3}{2}}$

**Solution :** Reasoning as in Example 2, let us try the **substitution** $t = x^2+2$.

We have $\frac{dt}{dx} = 2x$.

Thus, $f(x) = 2x (x^2 + 2)^{\frac{3}{2}} = t^{\frac{3}{2}} \frac{dt}{dx}$

Now, we know from the Table that an anti-derivative of

$$t^{\frac{3}{2}} \text{ is } \frac{t^{\frac{5}{2}}}{\frac{5}{2}} = \frac{2}{5}\, t^{\frac{5}{2}}$$

Thus, $t^{\frac{3}{2}} = \frac{d}{dt}\left(\frac{2\, t^{\frac{5}{2}}}{5}\right)$ ( Why ? )

By the **Chain Rule,** therefore,

$$2\,x\,(x^2+2)^{\frac{3}{2}} = \frac{d}{dt}\left(\frac{2\, t^{\frac{5}{2}}}{5}\right)\frac{dt}{dx}$$

$$= \frac{d}{dx}\left[\frac{2}{5}\,(x^2+2)^{\frac{5}{2}}\right]$$

Hence, an anti-derivative of $f(x)$ is $\frac{2}{5}\,(x^2+2)^{\frac{5}{2}}$

Thus, $\int 2\,\mathbf{x}\,(\mathbf{x}^2+2)^{\frac{3}{2}}\,d\mathbf{x} = \frac{2}{5}\,(\mathbf{x}^2+2)^{\frac{5}{2}} + \mathbf{c}$

In the next section, we formally give this method of finding anti-derivatives of certain functions.

## 30.2 The Method of Substitution

The method suggested by the examples of Section 30.1 is to express $f(x)$ as

$$f(x) = G'(t)\,\frac{dt}{dx}$$

where $t$ is a suitable function $\varphi$ of $x$ and obtain $G[\,\varphi(x)\,]$ as an anti-derivative of $f(x)$. We thus have a working technique which we state in the form of a theorem.

**Theorem 36 :** **(The Method of Substitution) If a function f (x) can be expressed as**

$$f(x) = g(t)\ \frac{dt}{dx}$$

**for some function $t = \varphi(x)$ so that an anti-derivative $G(t)$ of $g(t)$ is readily obtained, then**

$$G(t) = G[\varphi(x)]$$

**is an anti-derivative of f (x) i.e.,**

$$\int f(x)\ dx = \int g(t)\,dt = G[\varphi(x)] + c$$

**Proof** : The proof of the theorem is rather trivial. We have, by the **Chain Rule**,

$$\frac{d}{dx} G[\varphi(x)] = \frac{d}{dx}[G(t)]$$

$$= G'(t)\frac{dt}{dx} = g(t)\frac{dt}{dx}$$

$$= f(x)$$

**Remark** : There is no '*magic lamp*' *rule* to know which particular substitution would work in a given case. Only our experience and an eye on the **Table** will enable us to guess a possible and, hopefully, correct substitution.

We consider some examples.

**Example 1** : Integrate $\sin(ax+b)$, $a \neq 0$

**Solution** : Here an obvious **substitution** is

$$t = ax + b$$

Then, $\frac{dt}{dx} = a$

Thus, $\sin(ax+b) = \sin t = \frac{1}{a}\sin t\,\frac{dt}{dx}$

Hence, $\int \sin(ax+b)\ dx = \int \frac{1}{a}\sin t\,\frac{dt}{dx}\ dx = \frac{1}{a}\int \sin t\ dt$

$$= -\frac{1}{a}\cos t + c$$

$$= -\frac{1}{a}\cos(ax+b) + c$$

Thus,

$$\int \sin(ax + b)\,dx = -\frac{1}{a}\cos(ax + b) + c$$

[ Can you integrate sin $(ax + b)$ when $a = 0$ ? ]

**Example 2 :** Evaluate $\int (ax + b)^n\,dx$; $a \neq 0, n \neq -1$

**Solution :** We let $t = ax + b$

Then, $\frac{dt}{dx} = a$

Thus, $(ax + b)^n = t^n = \frac{1}{a}\, t^n\, \frac{dt}{dx}$

Hence, $\int (ax + b)^n\,dx = \frac{1}{a}\int t^n \frac{dt}{dx}\,dx$

Now, since $n \neq -1$, we know an anti-derivative of $t^n$ is $\frac{t^{n+1}}{n + 1}$

Thus, $\int (ax + b)^n dx = \frac{1}{a}\,\frac{(ax+b)^{n+1}}{n + 1} + c$

**Remark :** We have thus evaluated $\int (ax+b)^n dx$ for all values of $n$ except the value $n = -1$. We will see later, in this unit, how to evaluate $\int \frac{dx}{ax + b}$.

**Example 3 :** Obtain an integral of $\sin x\,(\cos x)^2$

**Solution : Substitute t = cos x.** Then, $\frac{dt}{dx} = -\sin x$

Hence, $\sin x\,(\cos x)^2 = -t^2\,\frac{dt}{dx}$

And, $\int \sin x\,(\cos x)^2\,dx = \int -t^2\,\frac{dt}{dx}\,dx = -\int t^2\,dt$

$$= -\frac{t^3}{3} + c$$

$$= -\frac{1}{3}(\cos x)^3 + c$$

**Alternatively,** we know that,

$$\sin x\,(\cos x)^2 = (\sin x\ \cos x)\ \cos x = \frac{1}{2}\sin 2x \cos x$$

$$= \frac{1}{4}(\sin 3x + \sin x)$$

Thus, $$\int \sin x\,(\cos x)^2\,dx = \frac{1}{4}\int \sin 3x\,dx + \frac{1}{4}\int \sin x\,dx$$

$$= -\frac{1}{4}\,\frac{\cos 3x}{3} - \frac{1}{4}\cos x + c \quad \textbf{(Why ?)}$$

$$= -\frac{1}{12}\cos 3x - \frac{1}{4}\cos x + c$$

[The reader is advised to verify that the two answers are the same.]

**Example 4** : Find the area under the curve $y = \sqrt{3x+4}$ above the $x$-axis between $x = 0$ and $x = 4$. Draw the region.

**Solution** : We recall that $y = \sqrt{3x+4}$ is the upper part of the parabola $y^2 = 3x + 4$ (See Fig. 30.2). We have to find the area of the shaded region. i.e.,

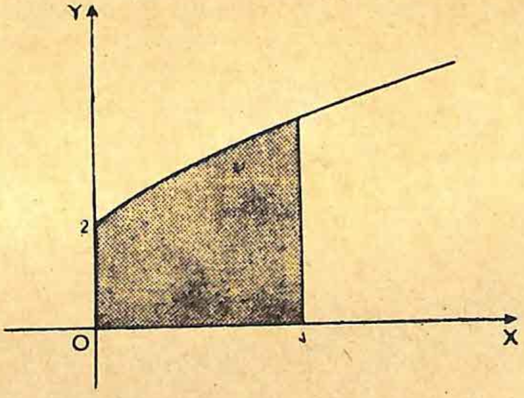

Fig. 30.2

we need to evaluate $\displaystyle\int_0^4 \sqrt{3x+4}\,dx$.

We recall from Example 2, Section 30.2 that an anti-derivative of $\sqrt{3x+4}$ is

$$\frac{1}{3}\,\frac{(3x+4)^{\frac{3}{2}}}{\frac{3}{2}} = \frac{2}{9}\,(3x+4)^{\frac{3}{2}}$$

$$\int_0^4 \sqrt{3x+4}\;dx = \frac{2}{9}\,(3x+4)^{\frac{3}{2}}\Big]_0^4$$

$$= \frac{2}{9}\left[16^{\frac{3}{2}} - 4^{\frac{3}{2}}\right] = \frac{2}{9}\,(64-8) = \frac{112}{9}$$

Thus, $\displaystyle\int_0^4 \sqrt{3x+4}\;dx = \frac{112}{9}$

**Example 5 .** Evaluate the definite integral

$$\int_{-1}^{2} \frac{x}{(x^2+1)^2}\,dx$$

**Solution :** We let $t = x^2+1$

Thus, $$\frac{dt}{dx} = 2x$$

And, $$\frac{x}{(x^2+1)^2} = \frac{1}{2}\cdot\frac{2x}{(x^2+1)^2} = \frac{1}{2\,t^2}\,\frac{dt}{dx}$$

We thus have

$$\int \frac{x}{(x^2+1)^2}\,dx = \int \frac{1}{2}\,t^{-2}\,dt = \frac{1}{2}\,\frac{t^{-2+1}}{-1} + c$$

$$= \frac{-1}{2t} + c = -\frac{1}{2\,(x^2+1)} + c$$

Hence, $$\int_{-1}^{2} \frac{x}{(x^2+1)^2}\, dx = -\frac{1}{2(x^2+1)}\Bigg]_{-1}^{2}$$

$$= \frac{-1}{10} - \left(\frac{-1}{4}\right) = \frac{3}{20}$$

**Alternatively**, we note that

$$when\ x = -1,\ t = x^2+1\Big]_{x=-1} = 2$$

$$when\ x = 2,\ t = x^2+1\Big]_{x=2} = 5$$

**Thus,**

$$\int_{-1}^{2} \frac{x}{(x^2+1)^2}\, dx = \frac{1}{2}\int_{2}^{5} \frac{1}{t^2}\, dt$$

$$= -\frac{1}{2t}\Bigg]_{2}^{5} = -\frac{1}{10} - \left(-\frac{1}{4}\right) = \frac{3}{20}$$

In the alternative evaluation of the integral we have made use of an important result which we state in the form of a theorem.

**Theorem 37 : If $f(x)$ can be written as $g(t)\,\frac{dt}{dx} = G'(t)\,\frac{t}{dx}$ for some function $t = \varphi(x)$ of x and if $\alpha = \varphi(a)$, $\beta = \varphi(b)$, [ i.e., when $x = a$, $t = \alpha$ and when $x = b$, $t = \beta$ ] then**

$$\int_{a}^{b} f(x)\, dx = \int_{\alpha}^{\beta} g(t)\, dt = G(\beta) - G(\alpha)$$

The proof is rather simple and is left to the reader as an exercise.

We consider some more examples.

**Example 6** : Evaluate $\displaystyle\int_{0}^{\frac{\pi}{2}} \sqrt{\cos\theta}\, \sin^3\theta\, d\theta$

**Solution** : We let $t = \cos\theta$

Thus, $\dfrac{dt}{d\theta} = -\sin\theta$

When $\theta = 0$, $t = \cos\theta\Big]_{\theta=0} = 1$

When $\theta = \dfrac{\pi}{2}$, $t = \cos\theta\Big]_{\theta=\frac{\pi}{2}} = 0$

$$\text{Hence, } \int_0^{\frac{\pi}{2}} \sqrt{\cos\theta}\ \sin^3\theta\, d\theta = -\int_1^0 \sqrt{t}\ (1-t^2)\, dt$$

$$= \int_0^1 \left( t^{\frac{1}{2}} - t^{\frac{5}{2}} \right) dt$$

$$= \int_0^1 t^{\frac{1}{2}}\, dt - \int_0^1 t^{\frac{5}{2}}\, dt \qquad \text{(Why ?)}$$

$$= \frac{2}{3}\ t^{\frac{3}{2}}\Big]_0^1 - \frac{2}{7}\ t^{\frac{7}{2}}\Big]_0^1$$

$$= \frac{2}{3} - \frac{2}{7} = \frac{8}{21}$$

$$\text{Thus, } \int_0^{\frac{\pi}{2}} \sqrt{\cos\theta}\ \sin^3\theta\, d\theta = \frac{8}{21}$$

**Example 7** : Find an integral of $\dfrac{(3\tan^2 x+2)\sec^2 x}{(\tan^3 x+2\tan x+9)^2}$

**Solution :** We have to evaluate $\displaystyle\int \frac{(3\tan^2 x+2)\sec^2 x}{(\tan^3 x+2\tan x+9)^2}\, dx$

**We first make the substitution t = tan x**

Then, $\frac{dt}{dx} = \sec^2 x$

Hence, $\int \frac{(3\tan^2 x+2)\sec^2 x}{(\tan^3 x+2\tan x+9)^2}\,dx = \int \frac{3t^2+2}{(t^3+2t+9)^2}\,dt$

How shall we evaluate this latter integral in $t$? **We make another substitution namely, we let**

$$\mathbf{u = t^3+2t+9}$$

$$\frac{du}{dt} = 3t^2 + 2$$

Hence, $\int \frac{(3t^2+2)}{(t^3+2t+9)^2}\,dt = \int \frac{du}{u^2} = -\frac{1}{u} + C$

Substituting $t^3+2t+9$ for $u$ and $\tan x$ for $t$, we have

$$\int \frac{(3\tan^2 x+2)\sec^2 x}{(\tan^3 x+2\tan x+9)^2}\,dx = -\frac{1}{\tan^3 x+2\tan x+9} + C$$

**Example 8 :** Evaluate the definite integral

$$\int_1^2 x\sqrt{3x-2}\,dx$$

**Solution :** Let us substitute $t = 3x - 2$

Then, $\frac{dt}{dx} = 3$

Hence, $x\sqrt{3x-2} = \frac{t+2}{3}\sqrt{t} = \frac{(t+2)\sqrt{t}}{3} \cdot \frac{1}{3} \frac{dt}{dx}$

$$= \frac{t^{\frac{3}{2}} + 2t^{\frac{1}{2}}}{9} \frac{dt}{dx}$$

**Also, when** $x=1$, $t=1$ **and when** $x=2$, $t=4$

Thus, $\int_1^2 x\sqrt{3x-2}\,dx=\frac{1}{9}\int_1^4 (t^{\frac{3}{2}}+2t^{\frac{1}{2}})\,dt$

$$=\frac{1}{9}\left[\frac{2}{5}t^{\frac{5}{2}}+\frac{4}{3}t^{\frac{3}{2}}\right]_1^4$$

$$=\frac{1}{9}\left[\frac{2}{5}(4^{\frac{5}{2}}-1)+\frac{4}{3}(4^{\frac{3}{2}}-1)\right]$$

$$=\frac{1}{9}\left[\frac{62}{5}+\frac{28}{3}\right]=\frac{326}{135}$$

i.e., $\int_1^2 x\sqrt{3x-2}\,dx=\frac{326}{135}$

## EXERCISE 30.1

**Evaluate each of the following integrals :**

1. $\int\frac{x}{\sqrt{2x^2+3}}\,dx$
2. $\int \cot^3 x\,\mathrm{cosec}^2 x\,dx$
3. $\int \cos^3(ax+b)\sin(ax+b)\,dx$
4. $\int x\sin^3(x^2)\cos x^2\,dx$
5. $\int \sec^4 x\tan x\,dx$
6. $\int x\sqrt{2x+3}\,dx$
7. $\int 4x^3\sqrt{5-x^2}\,dx$
8. $\int \cos^3 x\sin^4 x\,dx$
9. $\int\frac{dx}{\sqrt{3-4x}}$
10. $\int \frac{\cos x}{(1+\sin x)^{\frac{3}{2}}}\,dx$

Evaluate each of the following definite integrals :

11. $\int_0^{\frac{1}{2}}\frac{x}{\sqrt{1-x^2}}\,dx$
12. $\int_0^{\left(\frac{\pi}{2}\right)^{\frac{1}{3}}} x^2\sin(x^3)\,dx$

13. $\int_{4}^{8} x\sqrt[3]{x-4}\,dx$

14. $\int_{0}^{\frac{\pi}{4}} \frac{dx}{1+\cos 2x}$

15. $\int_{0}^{\frac{\pi}{2}} \cos^3 x\ (\sin x)^{\frac{1}{2}}\ dx$

16. $\int_{0}^{\frac{\pi}{2}} \sin^2 t\ dt$

17. $\int_{0}^{\frac{\pi}{2}} \cos^2 t\ dt$

18. $\int_{0}^{\frac{\pi}{2}} \sin^3 t\ dt$

19. $\int_{0}^{\frac{\pi}{2}} \cos^3 t\ dt$

20. $\int_{0}^{\frac{\pi}{2}} \sin^4 t\ dt$

21. $\int_{0}^{\frac{\pi}{2}} \cos^4 t\ dt$

22. $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos x\ dx}{(\cos\frac{x}{2} + \sin\frac{x}{2})^3}$

$\left[\textbf{Hint :}\ \ \cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2}\right]$

23. $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{\sqrt{1+\cos x}}{(1-\cos x)^{\frac{5}{2}}}\ dx$

24. Compare the areas under the curves $y=\sin x$ and $y=\sin 2x$ above the $x$-axis between $x=0$ and $\frac{\pi}{3}$. Also compare the areas between $x=0$ and $\frac{\pi}{4}$.

25. Compare the areas under the curves $\cos^2 x$ and $\sin^2 x$ above the $x$-axis between $x=0$ and $\pi$.

26. Find the area under the curve $y=\cos^3 x$ above the $x$-axis from 0 to $x$.

**27.** Find the area under the curve $y=x \sin x^2$ above the $x$-axis from $x=0$ to $\sqrt{\frac{\pi}{2}}$.

**28.** Find the area under the curve $y=\sqrt{6x+4}$ above the $x$-axis from $x=0$ to 2.

***29** Find the area of the region bounded by the x-axis and $y=\frac{1}{2}(2-3x-2x^2)$

***30.** Find the area under the graph of each of the following above the $x$-axis between the indicated ordinates:

(i) $y=\sqrt[3]{6-x}$, $x=-2$ and $x=5$

(ii) $y=\frac{3}{(1-2x)^3}$ $x=-4$ and $x=-1$

## 30.3 Integration by Parts

*How shall we integrate functions, such as, x sin x or x cos x ?* Neither the **Method of Substitution** nor the **Table** helps us.

We develop below a technique for integrating such functions.

Let us see if the rule for the derivative of a product of two functions can help us. We recall that if $f(x)$ and $g(x)$ are two differentiable functions, then

$$\frac{d}{dx}[f(x)\,g(x)] = f(x)\,g'(x) + f'(x)\,g(x)$$

Hence an anti-derivative of $f(x)\,g'(x)+f'(x)\,g(x)$ is $f(x)\,g(x)$.

Thus, $\int [f(x)\,g'(x) + f'(x)\,g(x)]\,dx = f(x)\,g(x) + C$

i.e., $\int [f(x)\,g'(x)\,dx + \int f'(x)\,g(x)\,dx = f(x)\,g(x) + C$ (1)

Now, let $f(x) = x$ and $g(x) = \sin x$

Then, $f(x)\,g'(x) = x \cos x$

And, $f'(x)\,g(x) = \sin x$

From (1), therefore,

$$\int x \cos x\,dx + \int \sin x\,dx = x \sin x + C$$

i.e., $\int x \cos x\,dx - \cos x = x \sin x + C$

**Thus, $\int x \cos x \, dx = x \sin x + \cos x + C$**

[The reader is advised to use (1) and evaluate $\int x \sin x \, dx$. Let $f(x) = x$ and $g(x) = \cos x$.]

We rewrite (1) as

$$\int f(x) g'(x) \, dx = f(x) g(x) - \int f'(x) g(x) \, dx \qquad (2)$$

This is the **Method of Integration by Parts.** The key is to *write the integrand as $f(x) g'(x)$ such that $\int f'(x) g(x) \, dx$ is easy to evaluate.* Then use (2) to find the required integral.

We consider some examples.

**Example 1 :** Integrate $x^2 \sin x$

**Solution :** We would like to write $x^2 \sin x$ as $f(x) g'(x)$ such that $\int f'(x) g(x) dx$ is easy to evaluate.

One choice is to let $f(x) = x^2$ and $g'(x) = \sin x$, so that $g(x) = -\cos x$. Also, $f'(x) = 2x$

We now have

$$\int x^2 \sin x \, dx = x^2 (-\cos x) - \int 2x (-\cos x) \, dx$$

$$= -x^2 \cos x + 2\int x \cos x \, dx$$

But we already know that $\int x \cos x \, dx = x \sin x + \cos x + C'$

Thus, $\int x^2 \sin x \, dx = -x^2 \cos x + 2x \sin x + 2 \cos x + C$

**Remark :** What if we choose to let $f(x) = \sin x$ and $g'(x) = x^2$, so that $g(x) = \frac{x^3}{3}$ ? Is it possible to use the above method and integrate $x^2 \sin x$ ?

**The right choice of f and g′ is, therefore, very important. Again, there is no** *'magic lamp' rule* **to decide which function to designate as f and which as g′. Only our experience and insight into the problem help us to make a** *'proper'* **choice.**

**Example 2 :** Evaluate $\int_0^{\frac{\pi}{2}} x^2 \sin x \, dx$

**Solution :** We know from Example 1, Section 30.3 that an anti-derivative of $x^2 \sin x$ is $-x^2 \cos x+2x \sin x+2 \cos x$. Let us denote this anti-derivative by $F(x)$. We have

$$F(x) = -x^2 \cos x+2x \sin x+2 \cos x$$

**The Second Fundamental Theorem of Integral Calculus** (*Theorem 34, Unit XXIX*) *tells us that if F is an anti-derivative of a continuous function f, defined on the closed interval* $a \leqslant x \leqslant b$, *then*

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Thus, $$\int_0^{\frac{\pi}{2}} x^2 \sin x\,dx = -x^2 \cos x+2x \sin x+2 \cos x \Big]_0^{\frac{\pi}{2}}$$

$$=\left(-\frac{\pi^2}{4} \cos \frac{\pi}{2} + \pi \sin \frac{\pi}{2} + 2 \cos \frac{\pi}{2}\right) - (2 \cos 0)$$

i. e., $$\int_0^{\frac{\pi}{2}} x^2 \sin x\,dx = \pi-2$$

We thus see that we can use the **Second Fundamental Theorem of Integral Calculus** and (2) to express certain definite integrals in terms of other definite integrals which are easy to compute. We state this result in the form of a theorem :

**Theorem 38 : (Method of Integration by Parts for Definite Integrals) Let f and g be two differentiable functions. Then,**

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big]_a^b - \int_a^b f'(x)\,g(x)dx \tag{3}$$

**Proof :** The proof is rather simple. We recall that $f(x)g(x)$ is an anti-derivative of

$$f(x)g'(x) + f'(x)g(x)$$

Thus, by the **Second Fundamental Theorem of Integral Calculus,**

$$\int_a^b [f(x)\, g'(x) + f'(x)\, g(x)]\, dx = f(x)\, g(x)\Big]_a^b$$

i.e.,
$$\int_a^b f(x)\, g'(x)\, dx + \int_a^b f'(x)\, g(x)\, dx = f(x)\, g(x)\Big]_a^b$$

Transposing, we get

$$\int_a^b f(x)\, g'(x)\, dx = f(x)\, g(x)\Big]_a^b - \int_a^b f'(x)\, g(x)\, dx$$

This completes the proof.

**Remark** : Let us consider a function $\varphi(x) = g(x) + \alpha$, where $\alpha$ is a constant. Then $\varphi'(x) = g'(x)$

By Theorem 38, therefore,

$$f(x)\, \varphi(x)\Big]_a^b - \int_a^b f'(x)\, \varphi(x)\, dx = \int_a^b f(x)\, \varphi'(x)\, dx$$

Substituting $\varphi'(x) = g'(x)$ and $\varphi(x) = g(x) + \alpha$, we have

$$\int_a^b f(x)\, g'(x)\, dx = f(x)\, [g(x) + \alpha]\Big]_a^b - \int_a^b f'(x)\, [g(x) + \alpha]\, dx$$

$$= f(x)\, g(x)\Big]_a^b + \alpha f(x)\Big]_a^b - \int_a^b f'(x)\, g(x)\, dx$$

$$- \alpha \int_a^b f'(x)\, dx$$

But, $$\alpha \int_a^b f'(x)\, dx = \alpha f(x) \Big]_a^b$$

Thus,

$$\int_a^b f(x)\, g'(x)\, dx = f(x)\, g(x) \Big]_a^b - \int_a^b f'(x)\, g(x)\, dx$$

It does not really matter if we replace $g$ by $\varphi$ in Theorem 38. We, therefore, have an **alternative form of the Method of Integration By Parts** which we state as a theorem. The proof is rather trivial and is omitted.

**Theorem 39 : Let f be differentiable and let h be a continuous function. Then**

$$\text{(i)} \quad \int_a^b f(x)\, h(x)\, dx = [f(x) \int h(x)\, dx] \Big]_a^b - \int_a^b f'(x)\, [\int h(x)\, dx]\, dx$$

$$\text{(ii)} \quad \int f(x)\, h(x)\, dx = f(x)\, [\int h(x)\, dx] - \int f'(x)\, [\int h(x)\, dx]\, dx$$

*It is convenient to remember (ii) in words.* We say

$$\int \underbrace{f(x)}_{\text{1st}}\ \underbrace{h(x)}_{\text{2nd}}\, dx = (\text{1st function}) \cdot (\text{integral of 2nd})$$

$$- \int (\text{derivative of 1st}) \cdot (\text{integral of 2nd function})\, dx \quad (4)$$

We consider some examples.

**Example 3 :** Evaluate $\displaystyle\int_2^4 \frac{x^2+x}{\sqrt{2x+1}}\,dx$

**Solution :** We let $x^2 + x$ be designated as the **1st function** and $\dfrac{1}{\sqrt{2x+1}}$ as the **2nd function.** We use Theorem 39 and have

$$\int_2^4 \frac{x^2+x}{\sqrt{2x+1}}\,dx = \left[(x^2+x)\int\frac{dx}{\sqrt{2x+1}}\right]_2^4 - \int_2^4 (2x+1)\left[\int\frac{dx}{\sqrt{2x+1}}\right]dx$$

$$\text{Now, } \int\frac{dx}{\sqrt{2x+1}} = \frac{(2x+1)^{-\frac{1}{2}+1}}{2.\,\frac{1}{2}} = \sqrt{2x+1}$$

Thus,

$$\int_2^4 \frac{x^2+x}{\sqrt{2x+1}}\,dx = (x^2+x)\,\sqrt{2x+1}\,\Big]_2^4 - \int_2^4 (2x+1)\,\sqrt{2x+1}\,dx$$

$$= 60 - 6\sqrt{5} - \int_2^4 (2x+1)^{\frac{3}{2}}\,dx$$

$$= 60 - 6\sqrt{5} - \left.\frac{(2x+1)^{\frac{5}{2}}}{2.\,\frac{5}{2}}\right]_2^4$$

$$= 60 - 6\sqrt{5} - \frac{1}{5}\left(9^{\frac{5}{2}} - 5^{\frac{5}{2}}\right)$$

$$= 60 - 6\sqrt{5} - \frac{243}{5} + 5\sqrt{5} = \frac{57}{5} - \sqrt{5}$$

$$\text{Hence, } \int_2^4 \frac{x^2+x}{\sqrt{2x+1}}\,dx = \frac{57}{5} - \sqrt{5}$$

**Example 4** : $\int_0^{\frac{\pi}{2}} \sin^4 x \, dx$

**Solution :** We write $\sin^4 x = (\sin^3 x) \times (\sin x)$ and designate **$\sin^3$x** as the **1st function** and **sin x** as the **2nd function**. We recall that $\int \sin x \, dx = -\cos x + C$. Thus,

$$\int_0^{\frac{\pi}{2}} \sin^4 x \, dx = \sin^3 x \, (-\cos x)\Big]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} 3 \sin^2 x \cos x \, (-\cos x) \, dx$$

$$= 0 + 3 \int_0^{\frac{\pi}{2}} \sin^2 x \cos^2 x \, dx$$

$$= 3 \int_0^{\frac{\pi}{2}} \sin^2 x \, (1 - \sin^2 x) \, dx$$

$$= 3 \int_0^{\frac{\pi}{2}} \sin^2 x \, dx - 3 \int_0^{\frac{\pi}{2}} \sin^4 x \, dx$$

Transposing $3 \int_0^{\frac{\pi}{2}} \sin^4 x \, dx$ to L.H.S., we have,

$$4 \int_0^{\frac{\pi}{2}} \sin^4 x \, dx = 3 \int_0^{\frac{\pi}{2}} \sin^2 x \, dx$$

But, $\sin^2 x = \dfrac{1-\cos 2x}{2}$

Thus, $$4\int_0^{\frac{\pi}{2}} \sin^4 x\, dx = \frac{3}{2}\int_0^{\frac{\pi}{2}} (1-\cos 2x)\, dx$$

$$= \frac{3}{2}\left[x - \frac{\sin 2x}{2}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{3}{2}\left[\left(\frac{\pi}{2} - \frac{\sin \pi}{2}\right) - \left(0 - \frac{\sin 0}{2}\right)\right]$$

$$= \frac{3\pi}{4}$$

i.e., $$4\int_0^{\frac{\pi}{2}} \sin^4 x\, dx = \frac{3\pi}{4}$$

Hence, $$\int_0^{\frac{\pi}{2}} \sin^4 x\, dx = \frac{3\pi}{16}$$

**Example 5 :** Show that $\int_0^{\frac{\pi}{2}} \sin^6\theta\; d\theta = \frac{5}{6}\int_0^{\frac{\pi}{2}} \sin^4\theta\; d\theta$. Hence, evaluate $\int_0^{\frac{\pi}{2}} \sin^6\theta\; d\theta$

**Solution :** We write $\sin^6\theta = (\sin^5\theta)(\sin\theta)$ and use Theorem 39. We have

$$\int_0^{\frac{\pi}{2}} \sin^6\theta\; d\theta = \sin^5\theta\,(-\cos\theta)\Big]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} (5\sin^4\theta\cos\theta)(-\cos\theta)\; d\theta$$

$$= 5 \int_0^{\frac{\pi}{2}} \sin^4\theta \cos^2\theta \, d\theta$$

$$= 5 \int_0^{\frac{\pi}{2}} \sin^4\theta \, (1-\sin^2\theta) \, d\theta$$

Thus, $$6 \int_0^{\frac{\pi}{2}} \sin^6\theta \, d\theta = 5 \int_0^{\frac{\pi}{2}} \sin^4\theta \, d\theta$$

i.e., $$\int_0^{\frac{\pi}{2}} \sin^6\theta \, d\theta = \frac{5}{6} \int_0^{\frac{\pi}{2}} \sin^4\theta \, d\theta$$

**But**, from Example 4, we know that $\int_0^{\frac{\pi}{2}} \sin^4\theta \, d\theta = \frac{3}{16} \pi$

**Hence**, $$\int_0^{\frac{\pi}{2}} \sin^6\theta \, d\theta = \frac{5}{6} \cdot \frac{3}{16} \pi = \frac{5\pi}{32}$$

**Example 6** : Integrate $(1+x) \cos x$

**Solution** : We have

$$\int \underbrace{(1+x)}_{\textbf{1st}} \underbrace{\cos x}_{\textbf{2nd}} \, dx = (1+x) \sin x - \int \sin x \, dx$$

Thus, $$\int (1+x) \cos x \, dx = (1+x) \sin x + \cos x + C$$

**Example 7** : Evaluate $\int (1+x^2)\cos 2x\,dx$

**Solution** : We have

$$\int \underbrace{(1+x^2)}_{\text{1st}}\ \underbrace{\cos 2x}_{\text{2nd}}\,dx = (1+x^2)\,\frac{\sin 2x}{2} - \int \frac{2x\sin 2x}{2}\,dx$$

$$= \frac{1}{2}(1+x^2)\sin 2x - \int \underbrace{x}_{\text{1st}}\ \underbrace{\sin 2x}_{\text{2nd}}\,dx$$

$$= \frac{1}{2}(1+x^2)\sin 2x - \left[x\left(\frac{-\cos 2x}{2}\right) + \int \frac{\cos 2x}{2}\,dx\right]$$

$$= \frac{1}{2}(1+x^2)\sin 2x + \frac{1}{2}\,x\cos 2x - \frac{1}{2}\int \cos 2x\,dx$$

$$= \frac{1}{2}(1+x^2)\sin 2x + \frac{1}{2}\,x\cos 2x - \frac{1}{2}\,\frac{\sin 2x}{2} + C$$

**Thus,**

$$\int (1+x^2)\cos 2x\,dx = \frac{1}{4}(2x^2+1)\sin 2x + \frac{1}{2}\,x\cos 2x + C$$

**Remark** : We observe that we had to integrate by parts twice to be able to evaluate the given integral.

## EXERCISE 30.2

Integrate each of the following functions :

1. $(2-x)\sin x$
2. $x^2\cos x$
3. $x^2\sin 2x$
4. $x\sin nx$
5. $x^3\cos x^2$
6. $x^3\sin x^2$.
7. $x^{2n-1}\cos x^n$

[**Hint** : Write $x^{2n-1}\cos x^n = x^n . x^{n-1}\cos x^n$. Take $f(x) = x^n$, $h(x) = x^{n-1}\cos x^n$]

8. $x^{2n-1}\sin x^n$.

9. $x^{\frac{4}{3}} - 3 \sin 3x + 2$

10. $\cos 5x + \cos 2x \sin 2x - x^6 + 1$

Evaluate each of the following :

11. $\int_0^{\frac{\pi}{2}} x^2 \cos 2x \, dx$

12. $\int_0^{\frac{\pi}{4}} (1 - x^2) \sin 2x \, dx$

13. $\int_0^{\left(\frac{\pi}{2}\right)^{\frac{1}{3}}} x^5 \sin x^3 \, dx$

14. $\int_0^{\frac{\pi}{15}} (x \cos 5x - 2 \sin 5x) \, dx$

15. $\int_0^{\frac{\pi}{6}} (2 + 3x^2) \cos 3x \, dx$

## 30.4. Logarithmic Functions and Their Derivatives

We are still not in a position to integrate functions, such as, $\frac{1}{x}$ or $\frac{1}{ax+b}$. In this section, we will study a function whose derivative is $\frac{1}{x}$.

### 30.4.1 Logarithmic Function

We give below a graph of the function

$$f(x) = \frac{1}{x}$$

for $x > 0$. (See Fig. 30.3)

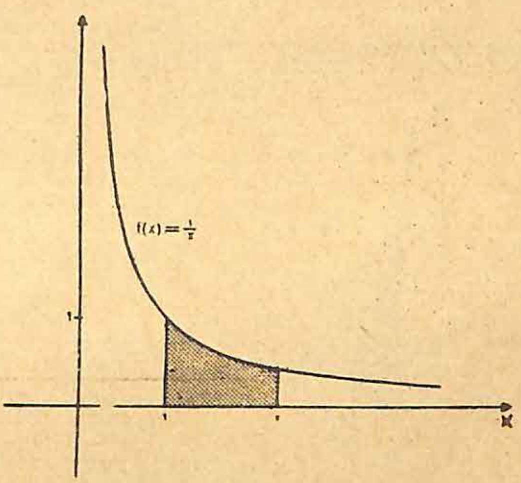

Fig. 30.3

We now make the following definition :

**Definition 13 : The natural logarithmic function*, ln, is defined as**

$$\ln x = \int_1^x \frac{1}{t}\,dt$$

**for $x > 1$. We note that $\ln 1 = 0$.**

We see that *ln x represents the area under the graph of* $y = \frac{1}{x}$ *between the ordinates at 1 and x, provided that* $x > 1$.

---

*'ln' is an abbreviated notation for 'log natural' or the 'natural logarithmic function'.

**For $0<x<1$.**

$$\ln x = -\int_x^1 \frac{1}{t}\,dt$$

i.e., *for $0<x<1$, $\ln x$ is the negative of the area under the graph of $y=\frac{1}{x}$ between the ordinates at $x$ and $1$.* (See Fig. 30.4)

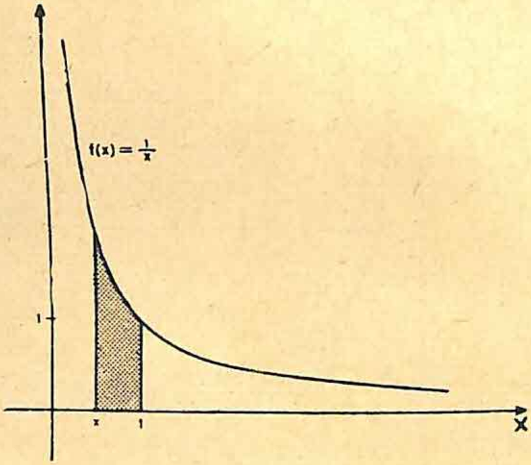

Fig. 30.4

We thus have the following situation:

$\ln x < 0$ for $0 < x < 1$

$\ln x = 0$ for $x = 1$

$\ln x > 0$ for $x > 1$

It may appear at first that we cannot say much about this new function because we do not know how to find an anti-derivative of $\frac{1}{t}$. However, as we shall see, the **First Fundamental Theorem** enables us to derive several beautiful properties of this function.

### 30.4.2 Properties of the Logarithmic Function

**PI.** $\frac{d}{dx}(\ln x) = \frac{1}{x}$

This follows directly from the **First Fundamental Theorem**, since $\frac{1}{x}$ is continuous for $x > 0$.

**Remark 1:** It follows immediately that $ln\, x$ is **an increasing function**, since its derivative, $\frac{1}{x}$, is always positive.

**Remark 2:** An immediate consequence of Remark 1 is that $ln\, x_1 \neq ln\, x_2$ if and only if $x_1 \neq x_2$.

**Remark 3:** It is clear that **ln x, for x >0, is a continuous function.** (Why ?)

**P II. ln (ab) = ln a+ln b, a > 0, b>0**

Let us first consider the case **when a > 1, b > 1**

$$\text{Then, } \ln (ab) = \int_1^{ab} \frac{1}{t}\, dt$$

$$= \int_1^{a} \frac{1}{t}\, dt + \int_a^{ab} \frac{1}{t}\, dt$$

$$= \ln a + \int_a^{ab} \frac{1}{t}\, dt \tag{1}$$

In (1), let us consider $\int_a^{ab} \frac{1}{t}\, dt$

Let us substitute $y = \frac{t}{a}$

Then, $\frac{dy}{dt} = \frac{1}{a}$

Now, when $t = a$, $y=1$ and when $t = ab$, $y=b$.

$$\text{Thus, } \int_a^{ab} \frac{1}{t}\, dt = \int_1^{b} \frac{1}{ay} \cdot a\, dy = \int_1^{b} \frac{1}{y}\, dy = \ln b$$

Substituting in (1), we get

$$\ln ab = \ln a + \ln b$$

[The reader is advised to furnish similar proof for other cases of $a$ and $b$.]

**P III.** $\ln\left(\frac{a}{b}\right) = \ln a - \ln b,\ a>0,\ b>0$

We write, $\ln a = \ln\left(\frac{a}{b}\cdot b\right)$

By P II, we have $\ln a = \ln\left(\frac{a}{b}\right) + \ln b$

Thus, $\ln\left(\frac{a}{b}\right) = \ln a - \ln b$

which proves the result.

**P IV** $\ln (a^n) = n \ln a$, **n a positive integer,** $a > 0$

[The proof of the property follows immediately by repeated application of P II and is left to the reader as an exercise.]

**P V.** $\ln\left(a^{\frac{p}{q}}\right) = \frac{p}{q}\ln a$, **where p and q are integers with** $q \neq 0$ **and** $a>0$

We prove this property for the case when p and q are positive integers.

We let $b = a^{\frac{1}{q}}$

Since $a>0$ and $q$ is a positive integer, it is clear that $b>0$

Now, $b^p = a^{\frac{p}{q}}$

Thus, $\ln\left(a^{\frac{p}{q}}\right) = \ln b^p = p \ln b$, by P IV $\quad (2)$

Also, $b^q = a$

Thus, $\ln a = q \ln b$, by P IV

Hence, $\ln b = \frac{\ln a}{q}$

Substituting in (2), we have

$$\ln\left(a^{\frac{p}{q}}\right)=\frac{p}{q}\ln a$$

[The proof of the case when $p$ and/or $q$ is negative is left to the reader as an exercise.]

Let us now see *how* ln $x$ *behaves for large values of* $x$.

Let us consider $x \geqslant 2^n$
Since ln $x$ is an increasing function, it follows that for $x \geqslant 2^n$

$$\ln x \geqslant n \ln 2$$

Now we can take $n$ as large as we want. Also $\ln 2 > 0$. It follows, therefore, that

$$\lim_{x\to\infty} \ln x=\infty$$

[We leave it as an exercise for the reader to show that

$$\lim_{x\to 0+} \ln x=-\infty$$

by considering $x \leqslant 2^{-n}$, etc.]

We can make use of the above properties and draw a graph of $y=\ln x$. (See Fig. 30.5)

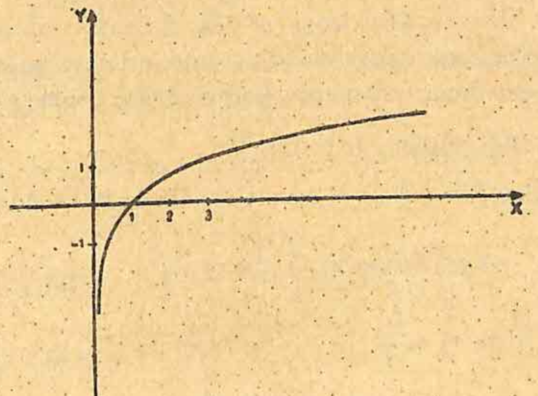

Fig. 30.5 : Graph of $y = \ln x, x > 0$

The graph lies entirely to the right of the $y$-axis since, *ln* $x$ is defined only for $x > 0$. Furthermore, it is a rising graph, crossing the $x$-axis at $x=1$ (because $\ln 1=0$)

Since $\lim_{x\to\infty} \ln x = \infty$, it is clear that the graph of $y=\ln x$ crosses any given horizontal line above the $x$-axis. Similarly, since $\lim_{x\to 0+} \ln x = -\infty$ the graph also crosses any given horizontal line below the $x$-axis.

Further, since $\ln x$ is continuous, it is intutively clear that $\ln x$ takes all real values. This means that given a real number 'b', there exists a positive real number '$a$' such that $\ln a=b$.

**Remark :** We have seen that $\ln x$ is defined for all $x>0$. **What if $x<0$?**

**When $x<0$, $\ln |x|$ has a meaning and we have**

$$\ln |x| = \ln(-x)$$

Thus, $\frac{d}{dx}(\ln |x|) = \frac{d}{dx}\ln(-x) = \frac{1}{x}$

We see, therefore, that **for all x, positive or negative,**

$$\frac{d}{dx}\ln |x| = \frac{1}{x}$$

Hence, $\int \frac{1}{x}\,dx = \ln |x| + C$

### 30.4.3 Derivative of Logarithmic Functions

We can now use our knowledge of the derivative of $\ln x$, the derivative of the composite functions and the derivative of the sum, product, quotient etc. to differentiate a large class of logarithmic functions. Let us consider some examples.

**Example 1 :** Differentiate $f(x)=(\ln x)^2$

**Solution :** Since $f(x)=(\ln x)^2$, we use the **Chain Rule** and have

$$f'(x)=2(\ln x)\frac{d}{dx}(\ln x)$$

But, $\frac{d}{dx}(\ln x) = \frac{1}{x}$

Thus, $f'(x) = 2(\ln x)\left(\frac{1}{x}\right)$

$$= \frac{2}{x}\ln x$$

**Example 2 :** Find the derivative of $x^2 \ln x$

**Solution :** Let $f(x)=x^2 \ln x$

We know how to find the derivative of the product. Thus we have

$$f'(x)=x^2\frac{d}{dx}(\ln x)+\ln x\frac{d}{dx}(x^2)$$

Now it is easy to see that

$$f'(x)=x^2\frac{1}{x}+(2x)\ln x$$

$$=x+2x\ln x$$

### 30.4.4 Logarithmic Functions and Integrals

*We would now like to find the area under the curve* $y=\ln x$ *between two given ordinates.* We, therefore, need an anti-derivative of $\ln x$.

Since $\ln x$ is continuous, an anti-derivative of **ln x exists**. Let us study the following example :

**Example 3 :** Compute $\int \ln x\, dx$

**Solution :** Since we do not know any anti-derivative of $\ln x$, we evaluate the integral by the method of **integration** by parts.

We write the integrand $\ln x$ as $(\ln x)(1)$ and designate 1 as the second function. We have

$$\int \ln x\, dx = \int (\ln x)\, 1\, dx = x\ln x - \int x\left(\frac{1}{x}\right)dx$$

$$= x\ln x - \int 1\, dx$$

$$= x\ln x - x + C$$

Hence, $\int \ln x\, dx = x\ln x - x + C$

We consider another example.

**Example 4 :** Evaluate $\int x \ln x \, dx$

**Solution :** We integrate by parts and have

$$\int x \ln x \, dx = (\ln x)\frac{x^2}{2} - \int \frac{x^2}{2}\left(\frac{1}{x}\right) dx$$

$$= \frac{x^2}{2} \ln x - \frac{1}{2}\int x \, dx$$

$$= \frac{x^2}{2} \ln x - \frac{1}{2}\left(\frac{x^2}{2}\right) + C$$

Thus, $\displaystyle\int x \ln x \, dx = \frac{x^2}{4}\,(2 \ln x - 1) + C$

It is remarkable that we can integrate certain trigonometric functions with the help of the logarithmic function. We consider some examples.

**Example 5 :** Compute $\int \tan x \, dx$

**Solution :** We have

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx$$

We let $t = \cos x$. Then, $\dfrac{dt}{dx} = -\sin x$

Thus, $\displaystyle\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx = -\int \frac{dt}{t} = -\ln | t | + C$

**Hence,[*] $\int \tan x \, dx = -\ln | \cos x | + C$**

**Example 6 :** Integrate $\dfrac{\sec^2 x}{\tan x + 2}$

**Solution :** We let $t = \tan x + 2$. Then, $\dfrac{dt}{dx} = \sec^2 x$

Thus $\displaystyle\int \frac{\sec^2 x}{\tan x + 2} \, dx = \int \frac{1}{t} \, dt$

$$= \ln | t | + C$$

Hence, $\displaystyle\int \frac{\sec^2 x}{\tan x + 2} \, dx = \ln | \tan x + 2 | + C$

---

[*] This integral is valid only in those intervals where $\cos x$ does not take the value 0.

**Remark** : We note that in each of the above two examples, the numerator is expressible as the derivative of the denominator. The **substitution, $t$=denominator,** helps us to compute the integral. This suggests the following important theorem :

**Theorem 40 : $\int \frac{\varphi'(x)}{\varphi(x)}\,dx = \ln|\varphi(x)| + C$ with the integral being valid in those intervals only, in which $\varphi(x)$ does not take the value zero.**

This theorem helps us to integrate functions which can be expressed in the form $\frac{\varphi'(x)}{\varphi(x)}$, for some function $\varphi(x)$. We consider some more examples. *However, we will not bother to state the domains of integration.*

**Example 7** : Integrate sec $x$.

**Solution** : We would like to be able to use Theorem 40. The '*trick*' is to multiply and divide the integrand by (sec $x$+tan $x$). We have

$$\int \sec x\,dx = \int \frac{\sec x(\sec x + \tan x)}{(\sec x + \tan x)}\,dx$$

Now, we see that

$$\frac{d}{dx}(\sec x + \tan x) = \sec x \tan x + \sec^2 x$$
$$= \sec x\,(\sec x + \tan x)$$

Thus, $\int \sec x\,dx = \int \frac{\varphi'(x)\,dx}{\varphi(x)}$

where $\varphi(x) = \sec x + \tan x$

By Theorem 40, therefore,

$$\int \sec x\,dx = \ln|\sec x + \tan x| + C$$

**Example 8** : Integrate $\frac{1}{ax+b}$, $a \neq 0$

**Solution :** We write $\frac{1}{ax+b} = \frac{1}{a}\left(\frac{a}{ax+b}\right)$

Since $\frac{d}{dx}(ax+b) = a$, Theorem 40 tells us that

$$\int \frac{dx}{ax+b} = \frac{1}{a}\ln|ax+b| + C$$

**Example 9 :** Integrate $\frac{x+3}{x^2+6x+4}$

**Solution :** We let $t = x^2+6x+4$

Then, $\frac{dt}{dx} = 2x+6 = 2(x+3)$

Thus, $\int \frac{x+3}{x^2+6x+4}\, dx = \frac{1}{2}\int \frac{dt}{t}$

$$= \frac{1}{2} \ln |t| + C$$

Hence, $\int \frac{x+3}{x^2+6x+4}\, dx = \frac{1}{2} \ln | x^2+6x+4 | + C$

**Example 10 :** Compute $\int \frac{\sin 2x - \sin x}{\sin^2 x + \cos x + 3}\, dx$

**Solution :** Since $\sin 2x = 2 \sin x \cos x$, we have

$$\frac{\sin 2x - \sin x}{\sin^2 x + \cos x + 3} = \frac{2 \sin x \cos x - \sin x}{\sin^2 x + \cos x + 3}$$

$$= \frac{\varphi (x)}{\varphi (x)},$$

where $\varphi(x) = \sin^2 x + \cos x + 3$

Thus, $\int \frac{\sin 2x - \sin x}{\sin^2 x + \cos x + 3}\, dx = \ln | \sin^2 x + \cos x + 3 | + C$

## EXERCISE 30.3

1. Prove that $\ln \left( a^{-\frac{p}{q}} \right) = -\frac{p}{q} \ln (a), a > 0, p > 0, q > 0.$

Find the derivative of each of the following :

2. $\ln 5x$
3. $(\ln x^2)^5$
4. $\ln (\sin x)$
5. $\ln (\cos x^2)$
6. $\sin x \ln x$
7. $(1+x^2) \ln 2x$
8. $(1-3x^5) \ln (\tan x)$
9. $\cos (\ln x)^2$

**10.** $x^2 \ln 5x - \ln(\cot x) + 5 \ln x$

**11.** $\dfrac{(\ln x)^2}{x}$

**12.** $\ln(x^2+5x+2) - \sin[\ln(x^3-1)] + 8$

Compute each of the following integrals ·

**13.** $\int x^2 \ln x \, dx$

**14.** $\int \ln(5+t) \, dt$

**15.** $\int x^3 \ln x \, dx$

**16.** $\int \dfrac{\ln x}{x} \, dx$

**17.** $\int \dfrac{\ln(x+2)}{(x+2)^2} \, dx$

**18.** $\int \dfrac{\ln x}{x^2} \, dx$

**19.** $\int \cot x \, dx$

**20.** $\int \dfrac{2\cos 2x + \sec^2 x}{\sin 2x + \tan x - 5} \, dx$

**21.** $\int \dfrac{\operatorname{cosec}^2 x + \sin x + 2x}{\cot x + \cos x - x^2} \, dx$

**22.** $\int \dfrac{6x-8}{3x^2-8x+5} \, dx$

**23.** $\int \dfrac{x^3-3x^2+x}{x^4-4x^3+2x^2-8} \, dx$

**24.** $\int \dfrac{\cos 2x + x + 1}{x^2 + \sin 2x + 2x} \, dx$

Find each of the following integrals :

**25.** $\int \ln(4-x) \, dx$

**26.** $\int x^3 \ln x \, dx$

**27.** $\int \dfrac{2x+9}{x^2+9x+20} \, dx$

**28.** $\int \tan 2x \, dx$

**29.** $\int \operatorname{cosec} x \, dx$

**[Hint :** Multiply the numerator and denominator by $(\operatorname{cosec} x - \cot x)$]

## 30.5 Integration of Rational Functions. Partial Fractions

So far, we have integrated polynomial functions and some simple trigonometric functions. As an isolated example, (Example 9, Section 30.4.4) we have also integrated **quotient of two polynomial functions**. Such a function is called a **rational function**.

**If the degree of the denominator is greater than the degree of the numerator, we call the function a proper rational function.** $\frac{2x+3}{x^2-2x+1}$ is a proper rational function, while $\frac{x^4+3x^2+1}{x^3-x^2+x+1}$ is not a proper rational function. **If the degree of the denominator** is, however, less than or equal to the degree of the **numerator,** we may use **long division** to express a rational function as the sum of a **polynomial function** and a **proper** rational function. Thus, $\frac{x^4+3x^2+1}{x^3-x^2+x+1}$ can be written as

$$(x+1)+\frac{3x^2-2x}{x^3-x^2+x+}$$

which is the sum of a polynomial function and a **proper** rational function.

We recall, from earlier classes, that we can add two **rational functions,** say,

$\frac{2}{x-3}$ and $\frac{3}{x-2}$ and write

$$\frac{2}{x-3}+\frac{3}{x-2}=\frac{5x-13}{(x-3)(x-2)}=\frac{5x-13}{x^2-5x+6}$$

$\frac{2}{x-3}$ and $\frac{3}{x-2}$ are called the **partial fractions** of $\frac{5x-13}{x^2-5x+6}$.

*In this section, we will develop some techniques that would enable us to write a given rational function as sum of its partial fractions and then learn how to integrate rational functions.*

We consider some examples.

Example 1 : Resolve $\frac{x^2-x-1}{x^3-x^2-6x}$ **into partial fractions.**

**Solution :** We note that the **given function is a proper** rational function. We proceed to factorize the denominator. We have

$$x^3-x^2-6x=x(x-3)(x+2)$$

Thus, $\frac{x^2-x-1}{x^3-x^2-6x}=\frac{x^2-x-1}{x(x-3)(x+2)}$

We write the given function as

$$\frac{x^2-x-1}{x(x-3)(x+2)}=\frac{A}{x}+\frac{B}{x-3}+\frac{C}{x+2} \tag{1}$$

where $A$, $B$ and $C$ are constants to be evaluated. **How shall we evaluate the constants ?** If we combine the fractions in the R.H.S. of (1), we get

$$\frac{A(x-3)(x+2)+Bx(x+2)+Cx(x-3)}{x(x-3)(x+2)}$$

Thus (1) can be rewritten as

$$\frac{x^2-x-1}{x(x-3)(x+2)}=\frac{A(x-3)(x+2)+Bx(x+2)+Cx(x-3)}{x(x-3)(x+2)} \tag{2}$$

We must choose $A$, $B$ and $C$, therefore, in such a way that the numerators in (2) are identical (since the denominators already tally). In other words, we must have

$$x^2-x-1=A(x-3)(x+2)+Bx(x+2)+Cx(x-3) \tag{3}$$

for all values of $x$, and hence, in particular, for specific values of $x$.

In (3), therefore, we let

$x=0$. Then, $-1=-6A$. Thus, $A=\frac{1}{6}$

$x=3$. Then, $5=15B$. Thus, $B=\frac{1}{3}$

$x=-2$. Then, $5=10C$. Thus, $C=\frac{1}{2}$

We substitute these values of A, B and C in (1) and have

$$\frac{x^2-x-1}{x^3-x^2-6x}=\frac{1}{6x}+\frac{1}{3(x-3)}+\frac{1}{2(x+2)}$$

**Example 2 :** Resolve $\frac{2x^4+x^3+3x^2+1}{2x^2+x-1}$ into partial fractions.

**Solution :** Since the degree of the numerator is greater than the degree of the denominator we use **long division** and have

$$\frac{2x^4+x^3+3x^2+1}{2x^2+x-1}=x^2+2-\frac{2x-3}{2x^2+x-1} \tag{1}$$

Now we proceed, as in Example 1, to resolve $\frac{2x-3}{2x^2+x-1}$ into partial fractions.

We have $2x^2+x-1=(2x-1)(x+1)$

Thus, we write

$$\frac{2x-3}{2x^2+x-1}=\frac{A}{2x-1}+\frac{B}{x+1} \qquad (2)$$

where A and B are constants.

We can rewrite (2) as

$$\frac{2x-3}{2x^2+x-1}=\frac{A(x+1)+B(2x-1)}{(2x-1)(x+1)} \qquad (3)$$

In the numerators of (3), we let

$x=-1$. Then, $-5=-3B$. Thus, $B=\frac{5}{3}$

$x=\frac{1}{2}$. Then, $-2=\frac{3}{2}A$. Thus, $A=-\frac{4}{3}$

Hence, (1) can be written as

$$\frac{2x^4+x^3+3x^2+1}{2x^2+x-1}=x^2+2+\frac{4}{3(2x-1)}-\frac{5}{3(x+1)}$$

**Example 3** : Resolve $\frac{2x^2+x+1}{(x-1)^2(x+2)}$ into partial fractions.

**Solution** : We write

$$\frac{2x^2+x+1}{(x-1)^2(x+2)}=\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{x+2} \qquad (1)$$

Hence, $2x^2+x+1=A(x-1)(x+2)+B(x+2)+C(x-1)^2 \qquad (2)$

In (2), we let

$x=1$. Then, $4=3B$. Thus, $B=\frac{4}{3}$

$x=-2$. Then, $7=9C$. Thus, $C=\frac{7}{9}$

$x=0$. Then, $1=-2A+2B+C$

Thus, $2\,A = 2\,B + C - 1 = \frac{8}{3} + \frac{7}{9} - 1 = \frac{22}{9}$

Hence, $A = \frac{11}{9}$

(1) can, therefore, be rewritten as

$$\frac{2x^2 + x + 1}{(x-1)^2\,(x+2)} = \frac{11}{9\,(x-1)} + \frac{4}{3\,(x-1)^2} + \frac{7}{9\,(x+2)}$$

### 30.5.1 Integration of Rational Functions

We have seen how to resolve rational function into partial fractions. We are thus in a position to integrate a very large class of rational functions. We consider some examples.

**Example 4 :** Evaluate $\int \frac{x+1}{x^2+2x+7}\,dx$

**Solution :** This rational function $\frac{x+1}{x^2+2\,x+7}$ can, of course, be integrated by simply letting

$$t = x^2 + 2\,x + 7$$

Then, $\frac{dt}{dx} = 2\,x + 2 = 2\,(x+1)$

Thus, $\int \frac{x+1}{x^2+2\,x+7}\,dx = \frac{1}{2} \int \frac{2\,(x+1)}{x^2+2x+7}\,dx$

Hence, $\int \frac{x+1}{x^2+2x+7}\,dx = \frac{1}{2} \ln \left| x^2 + 2\,x + 7 \right| + C$

**Example 5 :** Evaluate $\int \frac{x+1}{x^2+4x-5}\,dx$

**Solution :** We resolve the integrand into **partial fractions.** We have

$$\frac{x+1}{x^2+4x-5} = \frac{x+1}{(x+5)(x-1)} = \frac{A}{x+5} + \frac{B}{x-1} \qquad (1)$$

$$= \frac{A(x-1)+B(x+5)}{(x+5)(x-1)}$$

Thus, $x+1=A(x-1)+B(x+5)$ (2)

In (2), we let

$$x=1. \text{ Then } 2=6B. \text{ Thus, } B=\frac{1}{3}$$

$$x=-5. \text{ Then, } -4=-6A. \text{ Thus, } A=\frac{2}{3}$$

Hence (1) can be rewritten as

$$\frac{x+1}{x^2+4x-5}=\frac{2}{3(x+5)}+\frac{1}{3(x-1)}$$

And, $\displaystyle\int\frac{x+1}{x^2+4x-5}\,dx=\int\left[\frac{2}{3(x+5)}+\frac{1}{3(x-1)}\right]dx$

$$=\frac{2}{3}\int\frac{dx}{x+5}+\frac{1}{3}\int\frac{dx}{x-1}$$

i.e., $\displaystyle\int\frac{x+1}{x^2+4x-5}\,dx=\frac{2}{3}\ln|x+5|+\frac{1}{3}\ln|x-1|+C$

**Example 6 :** Evaluate $\displaystyle\int\frac{\sin\theta\cos\theta}{\cos^2\theta-\cos\theta-2}\,d\theta$

**Solution :** We first let $t=\cos\theta$

Then, $\dfrac{dt}{d\theta}=-\sin\theta$

Thus, $\displaystyle\int\frac{\sin\theta\cos\theta}{\cos^2\theta-\cos\theta-2}\,d\theta=-\int\frac{t\,dt}{t^2-t-2}$ (1)

We consider the integrand in the R.H.S. of (1). We have

$$\frac{t}{t^2-t-2}=\frac{t}{(t-2)(t+1)}=\frac{A}{t-2}+\frac{B}{t+1} \quad (2)$$

Thus, $t=A(t+1)+B(t-2)$ (3)

In (3), we let

$$t=-1. \text{ Then, } -1=-3B. \text{ Thus, } B=\frac{1}{3}$$

$$t=2. \text{ Then, } 2=3A. \text{ Thus, } A=\frac{2}{3}$$

Hence (2) can be rewritten as

$$\frac{t}{t^2-t-2} = \frac{2}{3(t-2)} + \frac{1}{3(t+1)}$$

(1) thus becomes

$$\int \frac{\sin\theta\cos\theta}{\cos^2\theta-\cos\theta-2}\,d\theta = -\frac{2}{3}\int\frac{dt}{t-2} - \frac{1}{3}\int\frac{dt}{t+1}$$

$$= -\frac{2}{3}\ln|t-2| - \frac{1}{3}\ln(t+1) + C$$

Thus,

$$\int \frac{\sin\theta\cos\theta}{\cos^2\theta-\cos\theta-2}\,d\theta = -\frac{2}{3}\ln|\cos\theta-2| - \frac{1}{3}\ln(\cos\theta+1) + C$$

**Example 7**: Integrate $\frac{2x^2+x+1}{(x-1)^2(x+2)}$

**Solution** : We refer to Example 3, Section 30.5 and have partial fractions of the given rational function as.

$$\frac{2x^2+x+1}{(x-1)^2(x+2)} = \frac{11}{9(x-1)} + \frac{4}{3(x-1)^2} + \frac{7}{9(x+2)}$$

Thus,

$$\int \frac{2x^2+x+1}{(x-1)^2(x+2)}\,dx = \frac{11}{9}\int\frac{dx}{x-1} + \frac{4}{3}\int\frac{dx}{(x-1)^2} + \frac{7}{9}\int\frac{dx}{x+2}$$

$$= \frac{11}{9}\ln|x-1| + \frac{4}{3}\,\frac{(x-1)^{-2+1}}{-1} + \frac{7}{9}\ln|x+2| + C$$

i.e., $$\int \frac{2x^2+x+1}{(x-1)^2(x+2)}\,dx = \frac{11}{9}\ln|x-1| - \frac{4}{3(x-1)} + \frac{7}{9}\ln|x+2| + C$$

## EXERCISE 30.4

Integrate each of the following :

1. $\dfrac{2x+5}{x^2-x-2}$

2. $\dfrac{12x+13}{6x^2+13x-63}$

3. $\dfrac{3x-1}{x^2-1}$

4. $\dfrac{5x-16}{5x^2-8x}$

5. $\dfrac{23}{6x^2+7x-20}$

6. $\dfrac{11x+6}{10x^2+11x+3}$

7. $\dfrac{26x+6}{8-10x-3x^2}$

8. $\dfrac{17x-2}{4x^2+7x-2}$

9. $\dfrac{x^3-x-2}{1-x^2}$

10. $\dfrac{2x^2+4x}{(2x+4)(x+1)(x+2)}$

11. $\dfrac{5x^3+18x^2-10x-6}{x(x+3)(5x-2)}$

12. $\dfrac{3x^2+2x-4}{(x^2-4)(x+1)}$

13. $\dfrac{12x^2-2x-9}{(4x^2-1)(x+3)}$

14. $\dfrac{x^4+x^3+2x^2+4x+1}{x(x+1)}$

15. $\dfrac{2x^3-3x^2-9x+1}{2x^2-x-10}$

16. $\dfrac{11x+8}{x^2(3x+8)}$

17. $\dfrac{2x}{(2x+1)^2}$

18. $\dfrac{4x^2+11x+11}{(2x+3)^2(x-2)}$

19. $\dfrac{5x^2+18x+17}{(x+1)^2(2x+3)}$

20. $\dfrac{4x^2-16x-3}{(3x+1)(2x-1)^2}$

21. $\dfrac{x^2+13x+19}{(x+2)^2(3x+5)}$

### 30.6 Exponential Functions and Their Derivatives

In this section, we will define yet another important function which is related t the natural logarithmic function. We will also discuss some properties of this ne function. But before we do so we need to develop the concept of inverse functions.

### 30.6.1 Inverse Functions

Let $y=f(x)=3x+1$ be a given function. We recall that $f(x)$ is an increasing function of $x$.

**What if we reverse the roles of x and y, i.e., if we write x as a function of y?** How would we do that? We solve for $x$ in terms of y. We have

$$x=\frac{y-1}{3}$$

i.e., we have a new function $g(x)=\frac{x-1}{3}$ [or $g(y)=\frac{y-1}{3}$, for that matter]. It is easy to verify that $g(x)$ is also an increasing function.

**We now ask a very natural question. What kind of relationship is there between the functions f and g?**

We note that

$$f[g(x)] = 3g(x)+1=3\left(\frac{x-1}{3}\right)+1$$

$$=x-1+1=x$$

i.e,, $\mathbf{f[g(x)]=x}$ (1)

Similarly, $g[f(x)]=\frac{f(x)-1}{3}=\frac{3x+1-1}{3}=x$

i.e., $\mathbf{g[f(x)]=x}$ (2)

We thus make the following statement:

**If f and g are two functions satisfying (1) and (2), then f is said to be the inverse of g and conversely. The example above tempts us to say that increasing functions have inverses.**

We draw the graphs of $f(x)=3x+1$ and $g(x)=\frac{x-1}{3}$ (See Fig. 30.6)

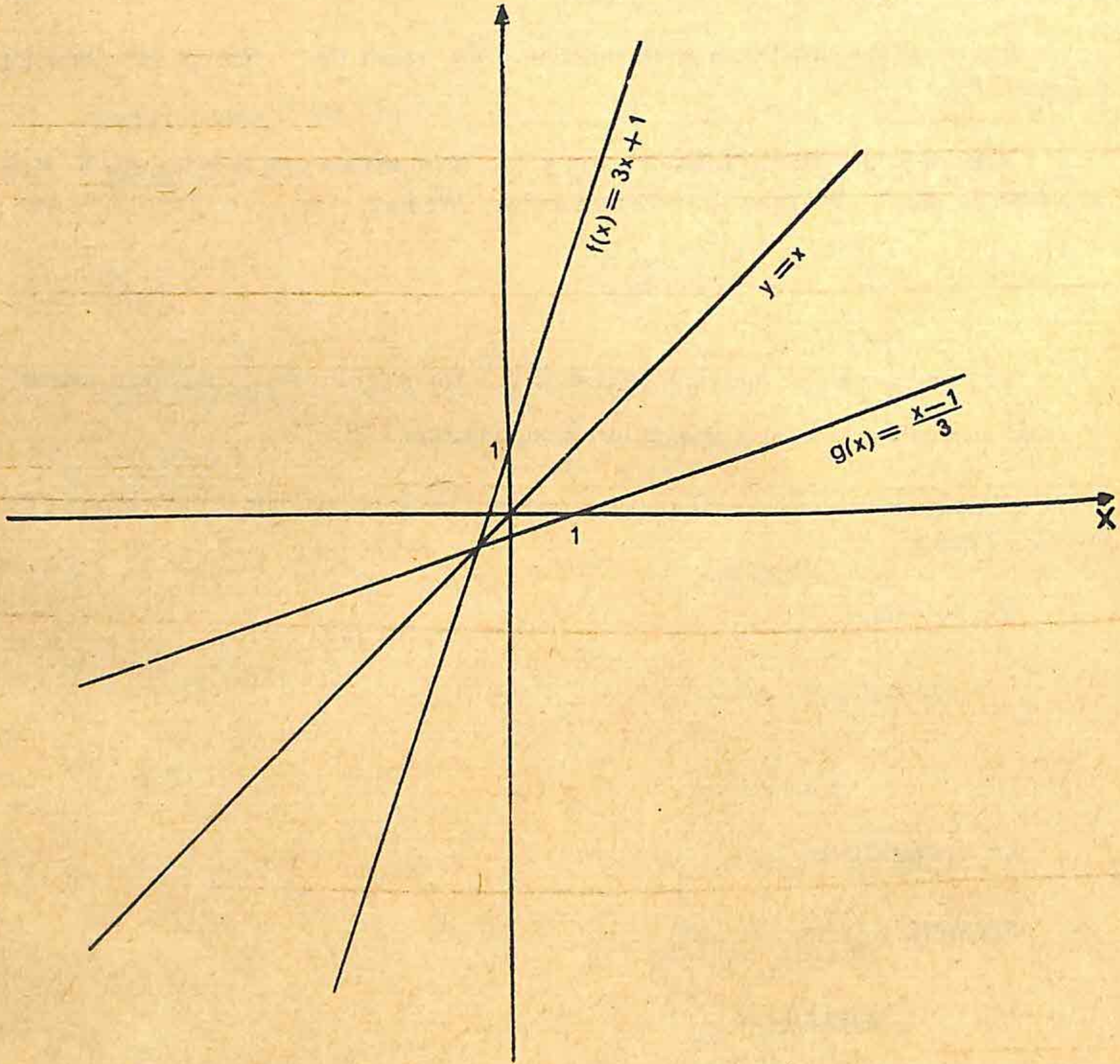

Fig. 30.6

We observe that **their graphs are mirror images of each in other the line y=x, i.e., g(x) can be obtained from f(x) by taking the image of f(x) in the line y=x and conversely.**

We consider some examples.

**Example 1** : Find the inverse function of $f(x) = \frac{2x+3}{4}$

**Solution :** We wish to find the function $g$ such that

$$g[f(x)] = x = f[g(x)] \tag{1}$$

We let $y = \frac{2x+3}{4}$. Solving for $x$, we get

$$4y = 2x+3, \text{ i. e., } x = \frac{4y-3}{2}$$

Thus, we let $g(y) = \frac{4y-3}{2}$

Hence, the inverse of $f(x) = \frac{2x+3}{4}$ is $g(x) = \frac{4x-3}{2}$

The reader may verify that $g(x)$ indeed satisfies (1).

### 30.6.2 Exponential Functions and their Derivatives

Let us examine the graph of $f(x) = \ln x$. (See Fig. 30.7) We recall that it is an increasing function. Thus, it is reasonable to expect that it has an inverse.

Let us take its image in the line $y=x$. What we get is the graph of an extremely important function, which we denote by $g(x) = \exp(x)$ and read as '**exponential function of x.**'

**The function exp (x) is defined for all real values of x. It is clear from the graph that it takes all positive real values.**

Further, this function satisfies (1) and (2).

i.e., $g[f(x)] = x$

**Thus,** $\exp(\ln x) = x, x > 0$

And, $f[g(x)] = x$. Thus, $\ln(\exp x) = x$ for all $x$

It follows, therefore, that $\ln[\exp(1)] = 1$ Usually, we write, $\exp(1) = e$. **Clearly,** $e > 1$.

Also, $\ln[\exp(0)] = 0$. But $\ln 1 = 0$. Hence, $\exp(0) = 1$.

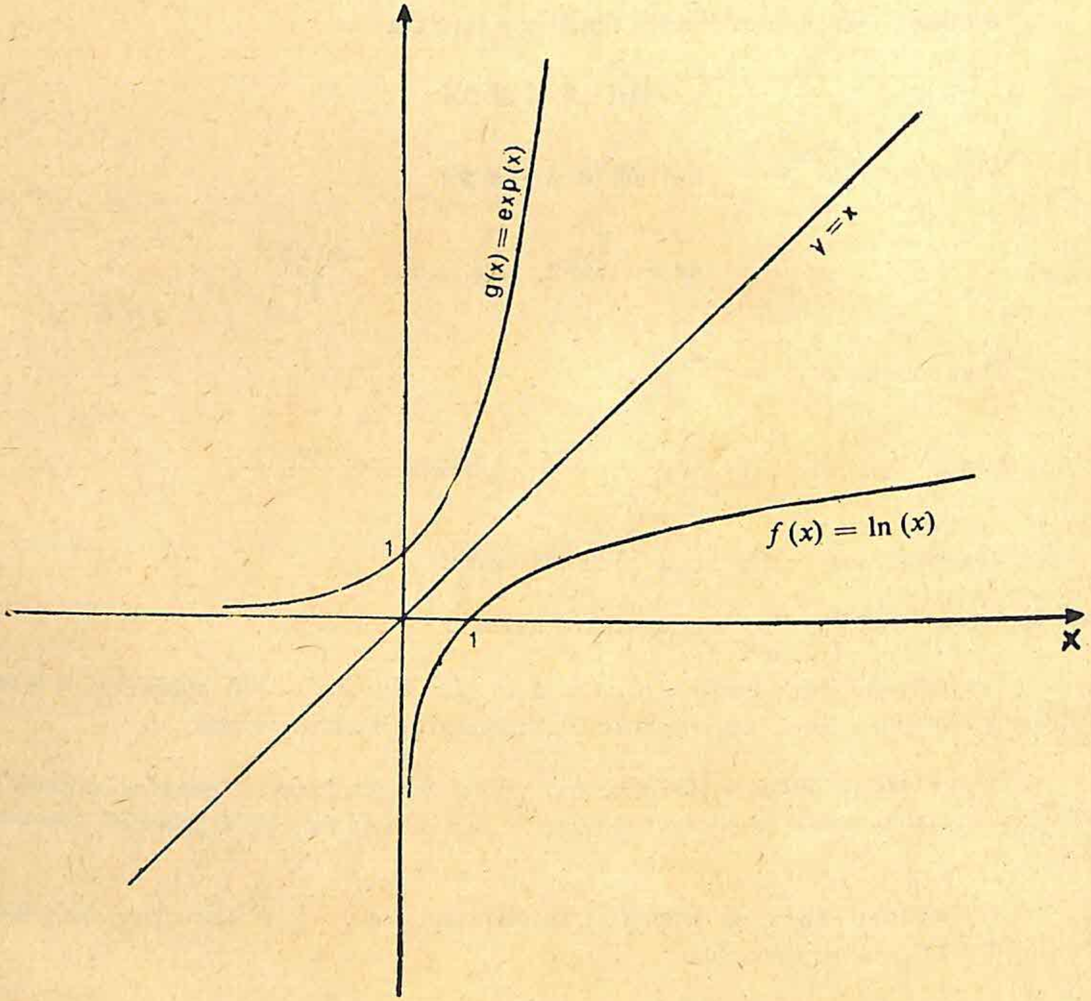

Fig. 30.7

We now have the following important theorem about the **derivative of the exponential function.**

**Theorem 41** : $\frac{d}{dx}$ **[exp (x)] = exp (x)**

i.e., **exp (x) is its own derivative**.

**Proof** : If we let

$$f(x) = \ln x \text{ and } g(x) = \exp(x),$$

then we recall that

$$g[f(x)] = x, \text{ for all } x > 0.$$

Using the rule for **the derivative of a composite function**, we get

$$g'[f(x)]\,f'(x)=1 \tag{1}$$

But, $f'(x)=\dfrac{1}{x}$, for all $x>0$

Thus, (1) gives us

$$g'[f(x)]\left(\frac{1}{x}\right)=1$$

i.e., $$g'[f(x)]=x \tag{2}$$

But, $g[f(x)]=x$. Thus (2) can be rewritten as

$$g'[f(x)]=g[f(x)] \tag{3}$$

Now, since $f(x)$ takes all real values when $x$ takes positive values, (3) gives us

$$g'(x)=g(x)$$

i.e., $$\frac{d}{dx}[\exp(x)]=\exp(x)$$

**Remark 1 :** An immediate consequence of this theorem is that **exp (x) is its own anti-derivative**

i.e., $$\int \exp(x)\,dx=\exp(x)+C$$

**Remark 2 :** Another consequence of the above theorem is that **exp (x) is an increasing function of x**. Moreover, **exp (x) → ∞ as x → + ∞** and **exp (x) → 0 as x → − ∞**

### 30.6.3 Properties of the Exponential Function

We give below some of the properties of the exponential function.

***PI** $\exp(x+y)=\exp(x)\exp(y)$; **for all x and y**

**PII** $\exp(nx)=[\exp(x)]^n$; **for all x and n a positive integer**

***Proof of PI :**

Since, $$\begin{aligned}\ln[\exp(x+y)]&=x+y\\&=\ln[\exp(x)]+\ln[\exp(y)]\\&=\ln[\exp(x)\exp(y)]\end{aligned}$$

Thus, $$\exp(x+y)=\exp(x)\exp(y)$$

**PIII** $\exp\left(\frac{x}{m}\right) = [\exp(x)]^{\frac{1}{m}}$; **for all x and m positive integers**

**PIV** $\exp\left(\frac{p}{q}x\right) = \left[\exp(x)\right]^{\frac{p}{q}}$; **for all x with p and q as positive integers and q $\neq$ 0.**

**PV** $\exp(-x) = \frac{1}{\exp(x)}$

**PVI** **exp (x) is never zero.**

**PVII** $\exp\left(\frac{p}{q}x\right) = [\exp(x)]^{\frac{p}{q}}$; **for p and q as integers and q $\neq$ 0**

**PVIII** $\exp\left(\frac{p}{q}\right) = e^{\frac{p}{q}}$

[The reader is advised to furnish a proof of each of the above properties on the lines of the proof given for PI.]

It is clear from the above properties that **the exponential function coincides with powers of e for rational numbers.**

Also, by Theorems 41 and 15, **exp (x) is continuous for all x.**

Thus, **exp (x) can be thought of as** $e^x$. PI through **PVIII** simply become, then, properties of surds.

We also have from Theorem 41 and Remark 1 that

$$\frac{d}{dx}(e^x) = e^x$$

And, $\int e^x dx = e^x + C$, **C a constant**

Using the exponential function and the logarithmic function, we can now define a whole class of new functions as follows :

**Definition 14**: **Assume a $>$ 0. We define the function $a^x$ as**

$$a^x = e^{x \ln a}$$

We note that

$$\frac{d}{dx}(a^x) = \frac{d}{dx}\left(e^{x \ln a}\right) = e^{x \ln a}\frac{d}{dx}(x \ln a)$$

$$= a^x \ln a$$

Thus, $\mathbf{\frac{d}{dx}\left(a^x\right) = a^x \ln a}$

An anti-derivative of $a^x \ln a$ is, therefore, $a^x$. In other words,

$$\int a^x\,dx = \frac{a^x}{\ln a} + C$$

We consider some examples.

**Example 2**: Find the derivative of

(i) $e^{x^2}$ (ii) $e^{x \sin x}$

**Solution**: (i) We let

$$f(x) = e^{x^2}$$

Thus, $f'(x) = e^{x^2}\frac{d}{dx}(x^2) = 2x\,e^{x^2}$

(ii) We let

$$y = e^{x \sin x}$$

**Then,** $\frac{dy}{dx} = e^{x \sin x}\frac{d}{dx}(x \sin x)$

i.e., $\frac{dy}{dx} = e^{x \sin x}(x \cos x + \sin x)$

**Example 3 :** Differentiate $e^{2x} \sin 2x$

**Solution :** We let

$$f(x) = e^{2x} \sin 2x$$

Then, $$f'(x) = e^{2x} \frac{d}{dx} (\sin 2x) + \sin 2x \frac{d}{dx} \; e^{2x})$$

$$= e^{2x} (2 \cos 2x) + 2 (\sin 2x) e^{2x}$$

$$= e^{2x} [ 2(\cos 2x + \sin 2x) ]$$

Thus, $$f'(x) = 2e^{2x} (\cos 2x + \sin 2x)$$

**Example 4 :** Find the derivative of

**(i)** $2^x$ **(ii)** $2^x \; e^{2(x+3)}$

**(iii)** $e^x \sin 3^x$

**Solution :** (i) We let

$$y = 2^x$$

Then, $$\frac{dy}{dx} = 2^x \ln 2$$

(ii) We let

$$f(x) = 2^x e^{2(x+3)}$$

Then, $$f'(x) = 2^x \frac{d}{dx}\left( e^{2(x+3)} \right) + e^{2(x+3)} \frac{d}{dx} \left(2^x\right)$$

$$= 2^x \left[e^{2(x+3)} \frac{d}{dx} [ 2(x+3) ]\right] + e^{2(x+3)} [2^x \ln 2]$$

$$= 2^x \left[e^{2(x+3)} (2)\right] + 2^x e^{2(x+3)} \ln 2$$

Thus, $f'(x) = 2^x e^{2(x+3)} (2+\ln 2)$

**(iii)** We let

$$y = e^x \sin 3^x$$

Then, $\dfrac{dy}{dx} = e^x (\cos 3^x) \dfrac{d}{dx}(3^x) + (\sin 3^x) \dfrac{d}{dx}(e^x)$

$$= e^x (\cos 3^x)\, 3^x \ln 3 + (\sin 3^x)\, e^x$$

i. e., $\dfrac{dy}{dx} = e^x \left[3^x (\cos 3^x) \ln 3 + \sin 3^x\right].$

**Example 5 :** If $y=x^x$ where $x$ is positive rational, find $\dfrac{dy}{dx}$

**Solution :** Clearly, the rule for the derivative of $x^n$ can not apply. We use PV of the natural logarithmic function and have

$$\ln y = \ln x^x = x \ln x$$

Thus, $\dfrac{d}{dx}(\ln y) = \dfrac{d}{dx}(x \ln x)$

i.e., $\dfrac{1}{y}\dfrac{dy}{dx} = x\left(\dfrac{1}{x}\right) + (\ln x)(1)$

$$= 1 + \ln x$$

Thus, $\dfrac{dy}{dx} = y\,(1 + \ln x)$

Or, $\mathbf{\dfrac{dy}{dx} = x^x\,(1+\ln x)}$

**Example 6 :** Integrate $x^2e^x$

**Solution :** We use the method of **integration by parts.** We recall that $\int e^x dx = e^x$. Thus

$$\int x^2e^x dx = x^2 e^x - \int 2x . e^x dx$$

$$= x^2 e^x - 2 \int xe^x dx$$

We integrate by parts once more and have

$$\int x^2e^x dx = x^2 e^x - 2 [ x e^x - \int 1. (e^x) dx]$$

$$= x^2 e^x - 2x e^x + 2e^x + C$$

Thus, $\int x^2 e^x dx = e^x (x^2 - 2x + 2) + C$

## EXERCISE 30.5

Find the inverse function of each of the following :

1. $f(x) = 2x+6$
2. $f(x) = \frac{3x-2}{4}$
3. $f(x) = \frac{x-1}{5}$

Differentiate each of the following :

4. $e^{x^3}$
5. $e^{(1-x^2)}$
6. $e^{x \cos x}$
7. $\sin x \, e^{\cos x}$
8. $\sin (e^x)$
9. $\sqrt{1-x^2} \, . \, e^{5x^2}$

**10.** $\ln x \,.\, e^{\tan x + x^2}$

**11.** $(\sin 2x + \cos 3x)e^{-5x}$

**12.** $e^{\sqrt{1-x^2}} \,.\, \tan x$

Find $\frac{dy}{dx}$ in each of the following :

**13.** $x^2e^y + \ln x = 0$

**14.** $ye^{x^2} + \tan x + \ln(\sin x) = 0$

**15.** $y^2e^y + e^{x^2} = 0$

**16.** $ye^x + 2^x \sin x = \cos y$

Integrate each of the following :

**17.** $(1+5x)\, e^{2x}$

**18.** $(1+x^2)\, e^x$

**19.** $x^3\, e^{5x}$

**20.** $x^2\, e^{-2x}$

**21.** $e^x \sin x$

**22.** $e^x \cos x$

**23.** $e^x \cos 3x$

**24.** $e^{5x} \cos x$

**25.** $2x^3e^{x^2}$

## MISCELLANEOUS EXERCISE VIII

### (On Units XXIX and XXX)

Find the area function for each of the following functions from the indicated first ordinate. Verify that the derivative of the area function is the function $f(x)$.

1. $f(x) = 5x-7, x = 4$
2. $f(x) = -6x+8, x = -7$
3. $f(x) = -4x-5, x = 6$

Find the area function for each of the following functions from the indicated first ordinate using **Archimedes method** Verify that the derivative of the area function is the function $f(x)$.

4. $x^2-5x, x = 3$
5. $2x^2, \quad x = 0$
6. $x^2-6, \quad x = 5$

Evaluate $\int_a^x f(x)\, dx$ for each of the following for the indicated value of $a$. Verify that $\frac{d}{dx}\left[ \int_a^x f(x)dx \right] = f(x)$

7. $f(x) = 5x-8, \quad a = 3$
8. $f(x) = -4x+2, a = -5$
9. $f(x) = -x+6, \quad a = 0$

**10.** Let $f(t) = t^2-t+1$. Evaluate $\int_x^6 f(t)dt$. Verify that

$$\frac{d}{dx}\left[\int_x^6 f(t)dt\right] = -f(x)$$

**Integrate** each of the following functions :

**11.** $3x^3+5 \sin x$

**12.** $\frac{2}{x^3}+\operatorname{cosec}^2 x-\frac{1}{x^4}$

**13.** $-3 \sin x+2 \cos x-\frac{1}{x^{\frac{5}{2}}}+3$

**14.** $\frac{x^2+x-1}{x^3}+4 \sin\frac{x}{2} \cos\frac{x}{2}$

Calculate the area under the graph of each of the following functions above the $x$-axis between the indicated ordinates:

**15.** $f(x) = 5x-8,\ x = 0,\ x = 10$

**16.** $f(x) = x^3-x^2+2x\ ;\ x = 1,\ x = 4$

**17.** $f(x) = 2\sqrt{x}+x^4-5\ ;\ x = 0,\ x = 1$

**18.** $f(x) = \cos^2\frac{x}{2}+\sin x\ ;\ x = 0,\ x = \pi$

**19.** $f(x) = (x^2+2)^2+2x\ ;\ x = 0,\ x = 2$

Evaluate each of the following :

**20.** $\int_1^4 \frac{x^3-5x^2+2}{\sqrt{x}}\, dx$

**21.** $\int_0^{\frac{\pi}{2}} (2 \sin x-5 \cos x+2)\, dx$

22. $\displaystyle\int_{0}^{\frac{\pi}{4}} (4\cos x - 6x^5 + \sin x)\, dx$

23. $\displaystyle\int_{0}^{\frac{\pi}{2}} \left(\sin\frac{x}{2}\cos\frac{x}{2} + x^2 - 1\right) dx$

24. $\displaystyle\int_{0}^{\frac{\pi}{2}} (x^2 - \sin x + \cos^2 x)\, dx$

25. $\displaystyle\int_{0}^{\pi} (\sin x + \cos^2 x + \sin^2 x - 1) dx$

Integrate each of the following:

26. $x^{\frac{1}{2}}\left(1 + x^{\frac{3}{2}}\right)^{\frac{1}{4}}$

27. $\dfrac{(x+2)^2}{\sqrt{x}}$

28. $\dfrac{x^3}{\sqrt{1+x^2}}$

29. $\dfrac{x^2}{\sqrt{1-x^3}}$

30. $\cos^2 3x$

31. $2x \sec^3 (x^2+3) \tan (x^2+3)$

32. $\operatorname{cosec}^4 x \cot x$

33. $\dfrac{x^2}{\sin^2 x^3}$

34. $\sin 4x \cos 3x$

35. $\dfrac{1}{x^3} \sin \dfrac{\pi}{x^2}$

36. $\dfrac{3x+5}{(3x^2+10x+2)^{\frac{3}{2}}}$

37. $x \sec x \tan x$

38. $x \sin 2x$

39. $x^3\sqrt{x^2+3}$

40. $x^5 \cos x^3$

41. $(2+7x)(\cos 6x)$

Evaluate each of the following :

42. $\displaystyle\int_{\sqrt{2}}^{2} \frac{2x\,dx}{(x^2-1)^{\frac{1}{2}}}$

43. $\displaystyle\int_{0}^{\sqrt{2}} \sqrt{2-x^2}\,dx$

44. $\displaystyle\int_{0}^{\frac{\pi}{2}} x^2 \cos x\,dx$

45. $\displaystyle\int_{0}^{1} (4x+3)^5\,dx$

46. $\displaystyle\int_{-1}^{1} \frac{x^5}{1+x^6}\,dx$

47. $\displaystyle\int_{0}^{\frac{\pi}{2}} \frac{\sin\theta}{\sqrt{1+\cos\theta}}\,d\theta$

48. $\int_{0}^{\frac{1}{2}} \frac{dx}{\sqrt{1-x}}$

49. $\int_{0}^{1} x^2 (1+x^3)^3 \, dx$

50. Find the area under the curve $y=x^2+7$ above the $x$-axis from $x=0$ and $x=2$.

51. Find the area between the curve $y=\sin 2x \cos 2x$ and the $x$-axis between $x=0$ and $x=\frac{\pi}{4}$

52. Find the area under the curve $y=\sin 2x+\cos 2x$ above the $x$-axis from 0 to $\frac{\pi}{4}$

53. Compare the areas under the curves $y=\sin x$ and $y=\cos x$ above the $x$-axis between 0 to $\frac{\pi}{4}$.

54. Compare the areas under the curves $\cos x$ and $\cos 2x$ above the $x$-axis between 0 and $\frac{\pi}{3}$.

55. Find the area under the graph of each of the curves above the $x$-axis between the given vertical boundaries:

(i) $y = 3+x-2x^2$ from $x = -1$ to $x = 0$

(ii) $y = x^3 + 6x^2 + 8x$ from $x = -4$ to $x = 0$

Differentiate each of the following :

56. $2 \ln \left(\frac{x-1}{x+1}\right)$

57. $x^2 \ln \frac{\sqrt{x^2+9}}{x+4}$

58 $\frac{\ln 2x}{x^2}$

**59.** $2 \ln x + \ln (\sin x^2) + \cos (\ln x)$

**60.** $\ln (\ln x)$

**61.** $(1 + x^2)\, e^{\sin x} + e^{-x \cos x} + \ln 3x$

**62.** $\ln (\sec x + \tan x)$

**63.** $\ln \sqrt{\dfrac{1-\cos x}{1+\cos x}}$

Find $\dfrac{dy}{dx}$ in each of the following :

**64.** $x \ln y + y \ln x = 0$

**65.** $x \ln x + e^{y} \sin x + 2x = 0$

**66.** $e^{y} \ln x + y \cos x + \tan(xy) = 0$

**67.** $x^2 e^{2y} + y^2 e^{2x} = 1$

**68.** $\ln (\sin x) + \cos (xy) + e^{xy} = 0$

Integrate each of the following :

***69.** $\sin (\ln x)$

***70.** $\sqrt{x} \ln 3x$

**71.** $\ln (\cos 2x) \sin 2x$

**72.** $x^2 e^{2x}$

**73.** $\dfrac{1-\sin x}{x+\cos x}$

**74.** $\dfrac{1}{x \ln x}$

**75.** $\dfrac{x}{\sqrt{a^2-x^2}}$

76. $\dfrac{2x^{\frac{1}{3}}}{1+x^{\frac{4}{3}}}$

77. $\dfrac{4\operatorname{cosec}^2 x}{1+\cot x}$

78. $\dfrac{\sec 2x \tan 2x}{\sqrt{e^{2\sec 2x}}}$

79. $(1-x^2)\ln x$

80. $e^{\cos^2 x}\sin 2x$

81. $\dfrac{\sqrt{2+\ln x}}{x}$

82. $\dfrac{e^{-\frac{1}{x}}+1}{x^2}$

83. $\dfrac{e^{4x}}{(1+e^{4x})^2}$

84. $(e^{x}+e^{-x})^2$

85. $x^2 e^{x^3}\cos e^{x^3}$

86. $\dfrac{x}{(x+2)(x-3)}$

87. $\dfrac{1}{x^2-5x+6}$

88. $\dfrac{x^2+8x+4}{x^3-4x}$

89. $\dfrac{x^2+5x-1}{x^3+x^2-6x}$

Find each of the following integrals :

90. $\displaystyle\int \frac{x^2}{x^2-1}\,dx$

91. $\displaystyle\int \ln x\,dx$

92. $\displaystyle\int \frac{\cos x}{1+\sin x}\,dx$

93. $\displaystyle\int_0^1 2e^{2x}\,(e^{2x}+3)\,dx$

94. $\displaystyle\int \frac{2x^2+5x+1}{x+1}\,dx$

95. $\displaystyle\int_0^1 (x+e^{-2x})\,(x^2-e^{-2x})^3\,dx$

96. $\displaystyle\int_1^3 (x^2+e^{x})(x^3+3e^{x}+4)\,dx$

97. $\displaystyle\int_0^2 \frac{e^{-x}-e^{x}}{5}\,dx$

## Some Important Results of Derivatives and Integrals

| $f(x)$ | $f'(x)$ |
|---|---|
| $K$ (constant) | $0$ |
| $x^n$ | $nx^{n-1}$ |
| $a\,g(x)$ | $a\,g'(x)$ |
| $g(x)+h(x)$ | $g'(x)+h'(x)$ |
| $\sum_{i=1}^{n} a_i\, f_i(x)$ | $\sum_{i=1}^{n} a_i\, f_i'(x)$ |
| $g(x)\,h(x)$ | $g(x)h'(x)+h(x)g'(x)$ |
| $\frac{g(x)}{h(x)}$ | $\frac{h(x)g'(x)-g(x)h'(x)}{[h(x)]^2}$ |
| $\sin x$ | $\cos x$ |
| $\cos x$ | $-\sin x$ |
| $\tan x$ | $\sec^2 x$ |
| $\cot x$ | $-\text{cosec}^2\, x$ |
| $\sec x$ | $\sec x \tan x$ |
| $\text{cosec}\, x$ | $-\text{cosec}\, x \cot x$ |
| $e^x$ | $e^x$ |
| $\ln \mid x \mid$ | $\frac{1}{x}$ |
| $a^x$ | $a^x \ln a$ |

| $f(x)$ | $\int f(x)dx$ |
|---|---|
| $K$ (constant) | $Kx$ |
| $x^n$ | $\frac{x^{n+1}}{n+1}$ $(n \neq -1)$ |
| $\frac{1}{x}$ | $\ln \mid x \mid$ |
| $\sin x$ | $-\cos x$ |
| $\cos x$ | $\sin x$ |
| $\tan x$ | $\ln \mid \sec x \mid = -\ln \mid \cos x \mid$ |
| $\cot x$ | $\ln \mid \sin x \mid$ |
| $\sec x$ | $\ln \mid \sec x + \tan x \mid$ |
| $\operatorname{cosec} x$ | $\ln \mid \operatorname{cosec} x - \cot x \mid$ |
| $\sec^2 x$ | $\tan x$ |
| $\operatorname{cosec}^2 x$ | $-\cot x$ |
| $\sec x \tan x$ | $\sec x$ |
| $\operatorname{cosec} x \cot x$ | $-\cot x$ |
| $e^x$ | $e^x$ |
| $a^x$ | $\frac{a^x}{\ln a}$ |
| $e^{bx}$ | $\frac{1}{b} e^{bx}$ |

# ANSWERS

## UNIT XXIV

### Exercise 24.1

**2.** Decreasing

### Exercise 24.2

**6.** Increasing in the interval $x \geqslant 0$, decreasing in the interval $x \leqslant 0$

**7.** Increasing in the interval $x \geqslant 0$, decreasing in the interval $x \leqslant 0$

**8.** Increasing in the interval $x \geqslant \frac{1}{6}$, decreasing in the interval $x \leqslant \frac{1}{6}$

**9.** Increasing in the interval $x \geqslant 2$ and $x \leqslant -2$, decreasing in the interval $-2 \leqslant x \leqslant 2$

**10.** Increasing in the interval $x \geqslant 1$, decreasing in the interval $x \leqslant 1$

**11.** Increasing in the interval $x \leqslant 0$, decreasing in the interval $x \geqslant 0$

### Exercise 24.4

**1.** 0

**2.** None

**3.** None

**10.** 2, 1

**11.** Not continuous

**12.** Continuous

**13.** 5

## UNIT XXV

### Exercise 25.1

**1.** 3

**2.** $\frac{1}{8}$

**3.** 1

**4.** 0

**5.** 7

**6.** 2 cos 2

**7.** 2

**8.** $\frac{5}{7}$

**9.** 0

**10.** 0

**11.** $-1$

**12.** 1

### Exercise 25.2

**1.** $-$ n $x$

**2.** $2 \cos 2x$

**3.** $-3 \sin 3x$

**4.** $4 \cos 4x$

5. $-5\sin 5x$

6. $x^{\frac{-2}{3}}(\cos 3x^{\frac{1}{3}})$

7. $\frac{3x^2\cos x^3}{2\sqrt{\sin x^3}}$

8. $4(x^2+\cos x)^3(2x-\sin x)$

9. $(x^3-2x+2)\cos x+(3x^2-2x-2)\sin x$

10. $\cos ax+\cos^4 bx\sin bx$

11. $4x(1-x^2)\sin(1-x^2)^2$

12. $\frac{2x\cos 2x-3\sin 2x}{x^4}$

13. $3\sin^2 x\sin 4x$

14. $m\sin^{m-1}x\sin(m+1)x$

15. $2x\sin(x^2+1)$

16. $3(3\sin^2 x-\cos^5 x+2)^2(3\sin 2x+5\cos^4 x\sin x)$

17. $5(t\sin t-\cos\sqrt{t})^4\times\left(t\cos t+\sin t+\frac{\sin\sqrt{t}}{2\sqrt{t}}\right)$

18. $\frac{\sin^2 u[\cos 3u(3\cos^2 u-\sin^2 u)\overset{3}{+2}\sin 2u\sin 3u]}{(\cos 3u)^2}$

19. 0

20. 1

21. $0 \quad 4\left(\frac{2-\pi}{4-\pi^2}\right)^2$

22. $\frac{3}{2}$

23. 0, 0

24. 0

25. $0,\ \frac{\pi^4}{1024\sqrt{2}}[20(\sqrt{2}+1)+\pi(2\sqrt{2}+1)]$

26. $6y+3\sqrt{3}x-3-\pi\sqrt{3}=0$

27 $2y-3\sqrt{3}=0$

28. $y-4=0$

29. $2y-4x+\pi=0$

30. $y-(2+\sqrt{2})x+(2+\sqrt{2})\frac{\pi}{4}-\sqrt{2}-1=0$

31. $6y-3x+\pi-3\sqrt{3}=0$

32. $2x-\pi=0$

33. $y+x-1=0$

34. $6(5-\sqrt{3})y+12x+63\sqrt{3}+15-8\pi=0$

35. $6x-\pi=0$

36. $(2+\sqrt{2})y+x-\frac{\pi}{4}-4-3\sqrt{2}=0$

37. $2x-\pi=0$

38. Increasing

39. Increasing

40. Increasing

41. Increasing

### Exercise 25,3

1. $\sec^2 x$

2. $-\operatorname{cosec}^2 x$

3. $\sec x\tan x$

4. $-\operatorname{cosec} x\cot x$

6. $-6x\cot^2 x^2\operatorname{cosec}^2 x^2$

**7.** $\frac{3}{2}\sqrt{x}\sec^2 x^{\frac{3}{2}}$

**8.** $-4\operatorname{cosec} 2x \cot 2x$

**9.** $\frac{2\sin 2x}{5}$

**10.** $\frac{3}{2}(x\cot^3 x)^{\frac{1}{2}}$
$\times[\cot^3 x - 3x\cot^2 x\operatorname{cosec}^2 x]$

**11.** $\sec^2(x+\frac{\pi}{4})$

**12.** $\frac{2\cos x}{(1-\sin x)^2}$

**13.** $x^3\tan\frac{x}{2}\sec^2\frac{x}{2}+3x^2\tan^2\frac{x}{2}$

**14.** $\frac{x^3\cos x+3x^2\sin x+\sec^2 x}{2\sqrt{x^3\sin x+\tan x}}$

**15.** $10(\tan^2 x+\operatorname{cosec}^2 x+2)^4$
$\times(\tan x\sec^2 x-\operatorname{cosec}^2 x\cot x)$

**16.** $2x\left(\cot x+\frac{\tan x}{1+x^2}\right)+\frac{x^2-1}{(1+x^2)^2}$
$\times[(1+x^2)\sec^2 x-(1+x^2)^2\operatorname{cosec}^2 x-2x\tan x]$

**17.** $(2x+1)\sec^2(x^2+x+2)$

**18.** $3a\tan^2(ax+b)\sec^2(ax+b)$

**19.** $\frac{3}{2}(\tan x+x\sec^2 x-x^2\cos x-2x\sin x$
$+\sec x\tan x)(\sqrt{x\tan x-x^2\sin x+\sec x})$

**20.** $\frac{1}{4}(-2x\operatorname{cosec} x^2\cot x^2+\frac{3}{2\sqrt{x}}\sin\sqrt{x}$
$+3\operatorname{cosec}^2 x^3-\frac{2\cot x^3}{x^3}$ ;
$\times(\operatorname{cosec} x^2-3\cos\sqrt{x}+\frac{1}{x^2}\cot x^3)^{\frac{-3}{4}}$

**21.** $\frac{8\tan x(\sec x+\tan x\sin x)}{(\tan x+\cos x)^3}$

**22.** $0,\ \frac{3\sqrt{2}\pi^4}{512}+\frac{(3\sqrt{2}+1)\pi^3}{32}+\frac{3\pi^2}{16}$

**23.** $\frac{8\sqrt{3}-36}{9}$

**24.** $\frac{3(32-13\sqrt{2})}{8}$

**25.** $\frac{y\sec^2 x+y\sin x+2}{2y\cos x-\tan x}$

**26.** $\frac{\sin 2x+y}{2\sin y-x}$

**27.** $\frac{y[2xy-y^2\cos(xy)-1]}{xy^2\cos(xy)+y^2-x}$

**28.** $\frac{(1+y^2)\sec x\tan x+y\operatorname{cosec}^2 x-2x}{\cot x-2y\sec x}$

**29.** $\frac{2xy-2(1+x^2)^2\tan x\sec^2 x}{(1+x^2)^2(4y+\cos y)+(1+x^2)}$

**30.** $\frac{\sec^2(x+y)+\sec^2(x-y)}{\sec^2(x-y)-\sec^2(x+y)}$

**31.** $-\frac{\sin(x+y)+y\cos x}{\sin(x+y)+\sin x}$

**32.** $\frac{y[1-\operatorname{cosec}^2(xy)]}{x\operatorname{cosec}^2(xy)-x+1}$

**33.** $2y-4x+\pi-2=0$

**34.** $3y-48\sqrt{3}x+16\sqrt{3}\pi-21=0$

**35.** $y-2\sqrt{\pi}\,x+\pi-4=0$

**36.** $4y+12x-27\pi-18=0$

**37.** $8y-4x+\pi-8=0$

**38.** $3y+(\pi^2+1)x-\pi(\pi^2+1)=0$

**39.** $24y-2x+\pi-96=0$

**40.** Increasing

**41.** Decreasing

## UNIT XXVI

### Exercise 26.1

**1.** 12.8 m, 4 seconds

**2.** 80

**3.** 1

**4.** $\frac{3}{4}$

**5.** $-2$

**6.** 0

**7.** 8, 8

**8.** 200, Rs 36500

**9.** $\frac{l}{2w}$ units from supporting end

**10.** 1, $\frac{59}{6}$

**11.** $\frac{\pi}{4}$

**12.** With radius $r=\left(\frac{50}{\pi}\right)^{\frac{1}{3}}$ cm

### Exercise 26.2

**1.** 8, $-10$

**2.** $-\frac{13}{2}$, $-11$

**3.** 4, $-\frac{89}{8}$

**4.** 1, $-5$

**5.** $\frac{47}{8}$, $\frac{89}{24}$

**6.** 17, $-111$

**7.** 144, 0

**8.** $3\sqrt[3]{3}$, 1

**9.** $\sqrt{2}$, i

**10.** $2\pi$, 0

**11.** $\sqrt{3}-\frac{\pi}{3}$, $-\frac{5\pi}{3}-\sqrt{3}$

**12.** 1, $\left(\frac{3}{4}\right)^{\frac{1}{3}}$

**13.** $\frac{\sqrt{6}}{8}$, 0

**14.** $\frac{3}{2}$, $-3$

**15.** 3, $\frac{1}{3}$

**17.** 2

**18.** 2500 sq. m

### Exercise 26.3

1. Maximum at $x = 1$ is 19 and minimum at $x = 3$ is 15
2. Maximum at $x = -2$ is 0 and minimum at $x = 0$ is $-4$
3. Minimum at $x = 2$ is 2
4. Maximum at $x = 0$ is $\frac{1}{2}$
5. None
6. Maximum at $x = \frac{2}{3}$ is $\frac{2\sqrt{3}}{9}$
7. Minimum at $x = \frac{\pi}{4}$ is $\frac{1}{2}$
8. Maximum at $x = \frac{\pi}{6}$ is $\frac{\sqrt{3}}{2} - \frac{\pi}{6}$

   and minimum at $x = \frac{\pi}{6}$ is $-\frac{\sqrt{3}}{2} + \frac{\pi}{6}$
9. Maximum at $x = 2$ is $\frac{1}{2}$
10. Maximum at $x = 2$ is $-1$
11. Maximum at $x = 1$ is 2
12. Minimum at $x = -\frac{7}{4}$ is $-\frac{3}{4\sqrt[3]{4}}$
13. $\frac{15}{2}$, $\frac{15}{2}$
14. The wire is to be cut at a distance of $\frac{28\pi}{\pi+4}$ m from one end

### Miscellaneous Exercise VI

5. Increasing in $0 \leqslant x \leqslant \frac{3\pi}{4}$ and $\frac{7\pi}{4} \leqslant x \leqslant 2\pi$, decreasing in $\frac{3\pi}{4} \leqslant x \leqslant \frac{7\pi}{4}$

14. Increasing in $x \geqslant \frac{1}{4}$

    decreasing in $x \leqslant \frac{1}{4}$
15. Increasing in $x \geqslant 0$, decreasing in $x \leqslant 0$
16. Increasing in $0 < x \leqslant 1$, decreasing in $x \geqslant 1$
17. Increasing in $x < -1$ and $x > -1$

22. $f(1) = 3$

26. $-2 \sin 5$
27. 0
28. 0
29. 1
30. Yes
31. $-2x \sin x^2$
32. $3 \cos 3x$
33. $2x \cos x^2 - \left[\frac{(1+x^2)\sec^2 x - 2x\tan x}{(1+x^2)^2}\right]$
34. $5 - 10 \operatorname{cosec}^2 x \cot x - 6x \operatorname{cosec}^2 x^2$

**35.** $3(\tan\sqrt{x}+x^2-\sin x)^2 \times\left(\frac{\sec^2\sqrt{x}}{2\sqrt{x}}+2x-\cos x\right)$

**36.** $-\operatorname{cosec} x \cot x+\operatorname{cosec}^2 x$

**37.** 2, −2

**38.** $\left(\frac{\pi}{2}-1\right)\sqrt{\frac{\pi}{2}}$

**39.** $\frac{1}{8\pi}(5\pi^2+56)(6+5\pi)$

**40.** $\frac{y\operatorname{cosec}^2x-y^3\sec^2x-\cos x}{3y^2\tan x+\cot x}$

**41.** $-\frac{y^2+\operatorname{cosec}^2x}{y\cos y+\sin y+2xy}$

**42.** $\frac{5}{(1-y^2)\cos y-2y\sin y+\sec^2 y}$

**43.** $y-(4+2\sqrt{3})x+\frac{\pi}{3}(4+2\sqrt{3})-2-\sqrt{3}=0$

**44.** $y-3=0,\ y+3=0$

**45.** $y=0,\ 2y+4x-\pi-2=0,$
$2y-4x+\pi+2=0$

**46.** $9y-6x+\pi-27\sqrt{3}=0$

**47.** $4y-8x+8-\pi=0,\ 4y+8x+8-\pi=0$

**48.** $x-2=0,\ x+2=0$

**49.** Maximum at $x=\frac{10+\sqrt{103}}{3}$, minimum at $x=3$

**50.** Minimum at $x=64$, maximum at $x=0$

**51.** Maximum at $x=9$, minimum at $x=\frac{5}{4}$

**52.** Maximum at $x=0$, minimum at $x=\frac{\pi}{2}$

**54.** $x=5$

**56.** 15, 15

**57.** Maximum at $x=1$ is 68, minimum at $x=-6$ is $-1647$ and minimum at $x=5$ is $-316$

**58.** Maximum at $x=2$ is 139, minimum at $x=2$ is 75

**59.** None

**60.** Minimum at $x=1$ is $-2$

**61.** Maximum at $x=8$ is 251 minimum at $x=0$ is $-5$

**62.** Minimum at $x=6$ is 14

**63.** Maximum at $x=22$ is $22\sqrt{11}$

**64.** Minimum at $x=1$ is 4

**65.** Minimum at $x=-\frac{\pi}{4}$ is $-\sqrt{2}$

**66.** $\frac{a}{2}$

**67.** $\frac{4ac-b^2}{4a}$

**68.** 27

**69.** 20

**70.** $(2\sqrt{3},3),\ (-2\sqrt{3},3)$

## UNIT XXVII

### Exercise 27.1

1. (*b*), (*c*) and (*d*) are not possible.

2. (*a*) $\frac{1}{2}$ (*b*) $\frac{2}{3}$ (*c*) $\frac{2}{3}$
(*d*) 1 (*e*) 0

3. (*a*) $\frac{1}{4}$ (*b*) $\frac{1}{2}$ (*c*) $\frac{1}{4}$

4. (*a*) $\frac{1}{13}$ (*b*) $\frac{1}{4}$ (*c*) $\frac{1}{52}$
(*d*) $\frac{3}{13}$ (*e*) $\frac{1}{26}$ (*f*) $\frac{1}{2}$

5. (*a*) HHH, HHT, HTH, HTT
THH, THT, TTH, TTT
(*b*) (*i*) $\frac{1}{8}$ (*ii*) $\frac{3}{8}$ (*iii*) $\frac{1}{2}$
(*iv*) $\frac{7}{8}$ (*v*) $\frac{1}{8}$

6. (*a*) $\frac{1}{16}$ (*b*) $\frac{1}{4}$
(*c*) $\frac{3}{8}$ (*d*) $\frac{15}{16}$

7. $\frac{5}{12}$

8. (*a*) $\frac{9}{20}$ (*b*) $\frac{7}{20}$ (*c*) $\frac{13}{20}$
(*d*) $\frac{11}{20}$ (*e*) $\frac{11}{20}$

9. (*a*) $S = \{11, 12, 13, 14, 16, 21, 22, 23, 24, 26, 31, 32, 33, 34, 36, 41, 42, 43, 44, 46, 61, 62, 63, 64, 66\}$
(*b*) (*i*) $\frac{1}{25}$ (*ii*) $\frac{2}{25}$ (*iii*) $\frac{4}{25}$

10. (*a*) $\frac{1}{12}$ (*b*) $\frac{1}{9}$ (*c*) $\frac{1}{18}$
(*d*) $\frac{1}{6}$ (*e*) $\frac{5}{18}$

### Exercise 27.2

1. (*a*) and (*g*) are mutually exclusive.

2. $\frac{1}{6}$

3. $\frac{11}{36}$

4. (*a*) $\frac{1}{26}$ (*b*) $\frac{4}{13}$
(*c*) $\frac{9}{13}$ (*d*) $\frac{1}{2}$

5. (*a*) 0.58 (*b*) 0.52 (*c*) 0.74

6. (*a*) $\frac{1}{36}$ (*b*) $\frac{5}{108}$ (*c*) $\frac{53}{54}$

7. (*a*), (*c*) and (*d*) are independent.

8. (*a*) $\frac{1}{36}$ (*b*) $\frac{1}{36}$ (*c*) $\frac{5}{18}$

9. $\frac{25}{1296}$

10. (*a*) $\frac{3}{8}$ (*b*) $\frac{1}{2}$ (*c*) $\frac{7}{8}$

11. (*a*) 0.18 (*b*) 0.12 (*c*) 0.42
(*d*) 0.28 (*e*) 0.72

12. $\frac{1}{338}$

# UNIT XXVIII

## Exercise 28.1

2 (*b*), (*c*) and (*e*) are not possible.

3.

| S : | 0 | 1 | 2 |
|---|---|---|---|
| P(S) : | $\frac{4}{9}$ | $\frac{4}{9}$ | $\frac{1}{9}$ |

4.

| X (Number of sixes) : | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| P(X) : | $\frac{125}{216}$ | $\frac{75}{216}$ | $\frac{15}{216}$ | $\frac{1}{216}$ |

5.

| X (Number of spades) : | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| P(X) : | $\frac{27}{64}$ | $\frac{27}{64}$ | $\frac{9}{64}$ | $\frac{1}{64}$ |

6.

| X (Number of doublets) : | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| P(X) : | $\frac{625}{1296}$ | $\frac{500}{1296}$ | $\frac{150}{1296}$ | $\frac{20}{1296}$ | $\frac{1}{1296}$ |

8.

| X (Number of red balls) : | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| P(X) : | $\frac{64}{343}$ | $\frac{144}{343}$ | $\frac{108}{343}$ | $\frac{27}{343}$ |

9.

| X (Number of face cards) : | 0 | 1 | 2 |
|---|---|---|---|
| P(X) : | $\frac{105}{221}$ | $\frac{96}{221}$ | $\frac{20}{221}$ |

10.

| X (Number of bad eggs) : | 0 | 1 | 2 |
|---|---|---|---|
| P(X) : | $\frac{12}{22}$ | $\frac{9}{22}$ | $\frac{1}{22}$ |

**13.** $\frac{13}{18}$

**14.** (*a*) $\frac{169}{625}$ (*b*) $\frac{312}{625}$

(*c*) $\frac{481}{625}$ (*d*) $\frac{144}{625}$

**15.** (*a*) $\frac{49}{121}$ (*b*) $\frac{16}{121}$ (*c*) $\frac{56}{121}$

**16.** (*a*) $\frac{1}{27}$ (*b*) $\frac{2}{9}$

(*c*) $\frac{26}{27}$ (*d*) $\frac{7}{27}$

**17.**

| Number of heads : | 0 | 1 | 2 |
|---|---|---|---|
| Probability : | $\frac{1}{4}$ | $\frac{1}{2}$ | $\frac{1}{4}$ |

**18.**

| Number of heads : | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| Probability : | $\frac{1}{8}$ | $\frac{3}{8}$ | $\frac{3}{8}$ | $\frac{1}{8}$ |

**19.**

| Number of heads : | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| Probability : | $\frac{1}{16}$ | $\frac{4}{16}$ | $\frac{6}{16}$ | $\frac{4}{16}$ | $\frac{1}{16}$ |

**20.**

| Total Number of points : | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|
| Probability : | $\frac{1}{36}$ | $\frac{2}{36}$ | $\frac{3}{36}$ | $\frac{4}{36}$ | $\frac{5}{36}$ | $\frac{6}{36}$ |

| Total Number of points : | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|
| Probability : | $\frac{5}{36}$ | $\frac{4}{36}$ | $\frac{3}{36}$ | $\frac{2}{36}$ | $\frac{1}{36}$ |

### Exercise 28.2

1. $\mu = 2$ , $\sigma^2 = 1$ , $\sigma = 1$

2. $\mu = 0$ , $\sigma^2 = 1.2$, $\sigma = 1.095$

3. $\mu = 1.6$, $\sigma^2 = 2.24$, $\sigma = 1.497$

4. $\mu = -0.2, \sigma^2 = 1.56$, $\sigma = 1.249$

5. $\mu = 1.1$, $\sigma^2 = 2.89$, $\sigma = 1.7$

6. $\frac{1}{2}$

7. $\mu = \frac{2}{13}$, $\sigma = 0.377$

8. $\frac{400}{2873}$

9. $\frac{4}{5}$

10. $\frac{4}{9}$

### Miscellaneous Exercise VII

1. $\frac{1}{8}$ , $\frac{3}{8}$ , $\frac{7}{8}$

2. $\frac{1}{2}$ , $\frac{1}{26}$

3. $\frac{3}{8}$

4. $\frac{1}{3}$

5. (*i*) $\frac{1}{2}$ (*ii*) $\frac{1}{2}$ (*iii*) $\frac{5}{6}$

6. $\frac{1}{3}$

7. (*i*) 0.60 (*ii*) 0.25 (*iii*) 0.50

8. $\frac{7}{18}$

9. $\frac{2}{3}$

10. (*a*) 0.55 (*b*) 0.65 (*c*) 0

11. $\frac{1}{2}$

12. (*a*) $\frac{2}{3}$ (*b*) $\frac{7}{9}$

13. (*a*) $\frac{2}{5}$ (*b*) $\frac{1}{2}$
(*c*) $\frac{1}{5}$ (*d*) $\frac{4}{5}$

14. $\frac{8}{27}$

15. $\frac{1}{36}$

16. (*a*) 0.2 (*b*) 0.2 (*c*) 0.3
(*d*) 0.3 (*e*) 0.7

17. (*a*) $\frac{36}{169}$ (*b*) $\frac{84}{169}$
(*c*) $\frac{120}{169}$ (*d*) $\frac{49}{169}$

18. $\frac{1}{4}$

19. (*a*) $\frac{1}{4}$ (*b*) $\frac{1}{169}$
(*c*) $\frac{1}{104}$ (*d*) $\frac{16}{169}$

20. (*a*) $\frac{1}{64}$ (*b*) $\frac{1}{4394}$ (*c*) $\frac{1}{4394}$

21. (*a*) $\frac{5}{144}$ (*b*) $\frac{1}{1014}$

22. $\frac{13}{30}$

23. $\frac{2}{27}$

24. (*a*) $\frac{27}{1000}$ (*b*) $\frac{999}{1000}$ (*c*) $\frac{729}{1000}$

25. (*a*) $\frac{11}{16}$ (*b*) $\frac{1}{16}$ (*c*) $\frac{1}{4}$

26. (*a*) $\mu=0.75$, $\sigma^2=3.9875$, $\sigma=2$.
    (*b*) $\mu=6.2$, $\sigma^2=5.76$, $\sigma=2.4$
    (*c*) $\mu=1.0$, $\sigma^2=4.10$, $\sigma=2.025$

27. $\mu=1.0$, $\sigma^2=\frac{2}{3}$, $\sigma=0.816$

28. $\frac{5}{9}$

29. $\mu=\frac{9}{7}$, $\sigma^2=\frac{36}{49}$, $\sigma=\frac{6}{7}$

30. $\mu=\frac{1}{2}$

## UNIT XXIX

### Exercise 29.1

1. $x^2-5x+6$
2. $x^2-5x$
3. $-\frac{x^2}{2}+6x+\frac{45}{2}$
4. $-\frac{x^2}{2}+6x-10$
5. $\frac{x^3}{3}-\frac{1}{3}$
6. $\frac{x^3}{3}+3x$
7. $\frac{x^3}{3}+\frac{x^2}{2}$
8. $\frac{x^4}{4}$

### Exercise 29.2

1. $x^2+x-30$
2. $x^2-5x$
3. $-\frac{x^2}{2}+6x-10$
4. $-\frac{3x^2}{2}+x+4$
6. $-2x+\frac{x^2}{2}$

### Exercise 29.3

1. $\frac{5x^3}{3}+6\sin x+c$
2. $-\frac{5}{x^2}-2\tan x+c$
3. $3x^2+\frac{5}{3}x^3-\frac{x^4}{2}-\frac{1}{x}+c$
4. $-6\cot x+\frac{1}{4x^4}+c$
5. $-2\cos x-\tan x-\frac{2}{\sqrt{x}}+c$
6. $\tan x-x-x^3+c$
7. $-\frac{4}{x}-\frac{4}{3}x^{\frac{3}{2}}-\cot x-x+c$
9. (*b*) 3 sec, 33.4 m/sec

**10.** $ax^2+bx+c,\ a,\ b,\ c\ \varepsilon\ R$

**11.** $-\frac{15}{2}$

**12.** 4

**13.** $\frac{143}{12}$

**14.** $\frac{\pi-2}{4}$

**15.** $\frac{2+\sqrt{2}}{3}$

**16.** $\frac{3}{5}\left(\frac{\pi}{3}\right)^{\frac{5}{3}}+\frac{\sqrt{3}}{2}$

**17.** $\frac{6136}{45}$

**18.** $\frac{2}{3}\ x^{\frac{3}{2}}-\frac{1}{2}\ x-\frac{\sin x}{2}+c$

**19.** $\frac{2}{5}\ x^{\frac{5}{2}}+\frac{\cos x}{2}+x+c$

**20.** $-3\cot x-\frac{5x^2}{2}-\cos x+c$

**21.** $\tan x+\sin x+c$

**22.** $\sin x-\cos x+\frac{2x^{\frac{7}{2}}}{7}+c$

**23.** $\frac{1}{t^2}-\frac{1}{t}-\frac{1}{3t^3}+\frac{7}{24}$

**24.** 7

**25.** $\frac{\pi^4}{1024}+\frac{\pi}{2}+2$

**26.** 0

**27.** $\frac{3x^{\frac{7}{3}}}{7}+2\cos x+2x+c$

**28.** $\tan x-\frac{1}{x}+\frac{5x^2}{2}+c$

**29.** $\sin x-\cot x-\frac{x^9}{9}+x+c$

**30.** $\frac{x^7}{7}-\sin x+\tan x+\frac{2x^{\frac{3}{2}}}{3}+c$

## UNIT XXX

### Exercise 30.1

**1.** $\frac{\sqrt{2x^2+3}}{2}+c$

**2.** $-\frac{\cot^4 x}{4}+c$

**3.** $-\frac{\cos^4(ax+b)}{4a}+c$

**4.** $\frac{\sin^4 x^2}{8}+c$

**5.** $\frac{\sec^4 x}{4}+c$

**6.** $\frac{(2x+3)^{\frac{5}{2}}}{10}-\frac{(2x+3)^{\frac{3}{2}}}{2}+c$

**7.** $\frac{20}{3}\ (5-x^2)^{\frac{3}{2}}+\frac{4}{5}\ (5-x^2)^{\frac{5}{2}}+c$

**8.** $\frac{\sin^5 x}{5}-\frac{\sin^7 x}{7}+c$

**9.** $-\frac{1}{2}\sqrt{3-4x}+c$

**10.** $-\frac{2}{\sqrt{1+\sin x}}+c$

**11.** $\frac{2-\sqrt{3}}{2}$

**12.** $\frac{1}{3}$

**13.** $\frac{132(4)^{\frac{1}{3}}}{7}$

**14.** $\frac{1}{2}$

**15.** $\frac{32}{65}$

**16.** $\frac{\pi}{4}$

**17.** $\frac{\pi}{4}$

**18.** $\frac{2}{3}$

**19.** $\frac{2}{3}$

**20** $\frac{3\pi}{16}$

**21.** $\frac{3\pi}{16}$

**22.** $\frac{2}{\cos\frac{\pi}{8}+\sin\frac{\pi}{8}}-\sqrt{2}$

**23.** $\frac{3}{2}$

**24.** $\frac{1}{2} : \frac{3}{4}, \quad \frac{2-\sqrt{2}}{2} : \frac{1}{2}$

**25.** $\frac{\pi}{2} : \frac{\pi}{2}$

**26.** $\sin x-\frac{1}{3}\sin^3 x$

**27.** $\frac{1}{2}$

**28.** $\frac{56}{9}$

**29.** $\frac{125}{48}$

**30.** (*i*) $\frac{45}{4}$ (*ii*) $\frac{2}{27}$

### Exercise 30.2

**1.** $-2\cos x+x\cos x-\sin x+C$

**2.** $x^2\sin x+2x\cos x-2\sin x+C$

**3.** $\frac{x\sin 2x}{2}-\frac{x^2\cos 2x}{2}+\frac{\cos 2x}{4}+C$

**4.** $\frac{\sin nx}{n^2}-\frac{x\cos nx}{n}+C$

**5.** $\frac{x^2\sin x^2}{2}+\frac{\cos x^2}{2}+C$

**6.** $\frac{\sin x^2}{2}-\frac{x^2\cos x^2}{2}+C$

**7.** $\frac{x^n\sin x^n}{n}+\frac{\cos x^n}{n}+C$

**8.** $\frac{\sin x^n}{n}-\frac{x^n\cos x^n}{n}+C$

**9.** $\frac{3x^{\frac{7}{3}}}{7}+\cos 3x+2x+C$

10. $\frac{\sin 5x}{5} - \frac{\cos 4x}{8} - \frac{x^7}{7} + x + C$

11. $-\frac{\pi}{4}$

12. $\frac{3}{4} - \frac{\pi}{8}$

13. $\frac{1}{3}$

14. $\frac{\pi\sqrt{3}}{150} - \frac{11}{50}$

15. $\frac{\pi^2 + 16}{36}$

### Exercise 30.3

2. $\frac{1}{x}$

3. $\frac{10}{x} (\ln x^2)^4$

4. $\cot x$

5. $-2x \tan (x^2)$

6. $\frac{\sin x}{x} + \cos x \ln x$

7. $\frac{1 + x^2}{x} + 2x \ln (2x)$

8. $2(1-3x^5) \operatorname{cosec} (2x) - 15x^4 \ln (\tan x)$

9. $-\frac{2}{x} \ln x \sin (\ln x)^2$

10. $x + 2x \ln (5x) + 2 \operatorname{cosec} (2x) + \frac{5}{x}$

11. $\frac{2 \ln x - (\ln x)^2}{x^2}$

12. $\frac{2x+5}{x^2+5x+2} - \frac{3x^2 \cos [\ln (x^3-1)]}{x^3-1}$

13. $\frac{x^3}{3} \ln x - \frac{x^3}{9} + C$

14. $t \ln (5+t) + 5 \ln (5+t) - t + C$

15. $\frac{x^4}{4} \ln x - \frac{x^4}{16} + C$

16. $\frac{(\ln x)^2}{2} + C$

17. $-\frac{1}{x+2}\left[\ln(x+2) + 1\right] + c$

18. $-\frac{1}{x} (\ln x + 1) + c$

19. $\ln | \sin x | + c$

20. $\ln | \sin(2x) + \tan x - 5 | + c$

21. $-\ln | \cot x + \cos x - x^2 | + c$

22. $\ln | 3x^2 - 8x + 5 | + c$

23. $\frac{1}{4} \ln | x^4 - 4x^3 + 2x^2 - 8 | + c$

24. $\frac{1}{2} \ln | x^2 + \sin(2x) + 2x | + c$

25. $(x-4)\ln(4-x) - x + c$

26. $\frac{x^4}{4} \ln x - \frac{x^4}{16} + c$

27. $\ln | x^2 + 9x + 20 | + c$

28 $\frac{1}{2} \ln | \sec 2x | + c$

29 $\ln | \operatorname{cosec} x - \cot x | + c$

### Exercise 30.4

1. $3 \ln|x-2| - \ln|x+1| + c$
2. $\ln|3x-7| + \ln|2x+9| + c$
3. $\ln|x-1| + 2\ln|x+1| + c$
4. $2\ln|x| - \ln|5x-8| + c$
5. $\ln|3x-4| - \ln|2x+5| + c$
6. $\frac{1}{2}\ln|2x+1| + \frac{3}{5}\ln|5x+3| + c$
7. $-\frac{5}{3}\ln|3x-2| - 7\ln|x+4| + c$
8. $4\ln|x+2| + \frac{1}{4}\ln|4x-1| + c$
9. $\frac{x^2}{2} - \ln|x+1| + \ln|x-1| + c$
10. $\ln|2x+4| - \ln|x+1| + \ln|x+2| + c$
11. $x + \ln|x+3| - \ln|5x-2| + \ln|x| + c$
12. $\ln|x+2| + \ln|x-2| + \ln|x+1| + c$
13. $\frac{1}{2}\ln|2x+1| - \frac{1}{2}\ln|2x-1| + 3\ln|x+3| + c$
14. $\frac{x^3}{3} + 2x + \ln|x+1| + \ln|x| + c$
15. $\frac{x^2}{2} - x + \ln|x+2| - \ln|2x-5| + c$
16. $\ln|x| - \frac{1}{x} - \ln|3x+8| + c$
17. $\frac{1}{2(2x+1)} + \frac{1}{2}\ln|2x+1| + c$
18. $\frac{1}{2(2x+3)} - \ln|x-1| + c$
19. $\frac{5}{2}\ln|2x+3| - \frac{4}{x+1}$
20. $\frac{1}{3}\ln|3x+1| - \frac{2}{2x-1}$
21. $\frac{1}{3}\ln|3x+5| - \frac{3}{x+2}$

### Exercise 30.5

1. $\frac{x-6}{2}$
2. $\frac{4x+2}{3}$
3. $5x+1$
4. $3x^2 e^{x^3}$
5. $-2xe^{(1-x^2)}$
6. $(\cos x - x\sin x)e^{x\cos x}$
7. $\cos x\, e^{\cos x} - \sin^2 x\, e^{\cos x}$
8. $e^{x} \cos(e^{x})$
9. $10x\sqrt{1-x^2}\; e^{5x^2} - \frac{x\, e^{5x^2}}{\sqrt{1-x^2}}$
10. $\frac{e^{\tan x + x^2}}{x} + \ln x(\sec^2 x + 2x)\, e^{\tan x + x^2}$

**11.** $\left[2\cos(2x)-3\sin(3x)\right]e^{-5x}$
$-5\left[\sin(2x)+\cos(3x)\right]e^{-5x}$

**12.** $-\frac{x\ e^{\sqrt{1-x^2}}}{\sqrt{1-x^2}}\tan x+\sec^2 x.e^{\sqrt{1-x^2}}$

**13.** $-\frac{2x^2e^{y}+1}{x^3\ e^{y}}$

**14.** $-\frac{2xy\ e^{x^2}+\sec^2 x+\cot x}{e^{x^2}}$

**15.** $-\frac{2x\ e^{x^2}}{2y\ e^{y}+y^2e^{y}}$

**16.** $-\frac{y\ e^{x}+2^{x}\sin x\ln 2+2^{x}\cos x}{e^{x}+\sin y}$

**17.** $\frac{5x\ e^{2x}}{2}-\frac{3\ e^{2x}}{4}+c$

**18.** $e^{x}(x^2-2x+3)+c$

**19.** $e^{5x}\left(\frac{x^3}{5}-\frac{3x^2}{25}+\frac{6x}{125}-\frac{6}{625}\right)+c$

**20.** $-\frac{e^{-2x}}{4}(2x^2+2x+1)+c$

**21.** $\frac{1}{2}\ e^{x}(\sin x-\cos x)+c$

**22.** $\frac{1}{2}\ e^{x}(\sin x+\cos x)+c$

**23.** $\frac{e^{x}}{10}(3\sin 3x+\cos 3x)+c$

**24.** $\frac{e^{5x}}{26}(\sin x+5\cos x)+c$

**25.** $(x^2-1)\,e^{x^2}+c$

## Miscellaneous Exercise VIII

**1.** $\frac{5x^2}{2}-7x-12$

**2.** $-3x^2+8x+203$

**3.** $-2x^2-5x+102$

**4.** $\frac{x^3}{3}-\frac{5x^2}{2}+\frac{27}{2}$

**5.** $\frac{2x^3}{3}$

**6.** $\frac{x^3}{3}-6x-\frac{35}{3}$

**7.** $\frac{5x^2}{2}-8x+\frac{3}{2}$

**8.** $-2x^2+2x+60$

**9.** $-\frac{x^2}{2}+6x$

**10.** $60-\frac{x^3}{3}+\frac{x^2}{2}-x$

**11.** $\frac{3x^4}{4}-5\cos x+c$

**12.** $-\frac{1}{x^2}-\cot x+\frac{1}{3x^3}+c$

**13.** $3\cos x+2\sin x+\frac{2}{3x^{\frac{3}{2}}}+3x+c$

**14.** $\ln|x|-\frac{1}{x}+\frac{1}{2x^2}-2\cos x$

15. 170

16. $\frac{231}{4}$

17. $-\frac{52}{15}$

18. $\frac{\pi+4}{2}$

19. $\frac{436}{15}$

20. $-\frac{152}{7}$

21. $\pi-3$

22. $1+\frac{3\sqrt{2}}{2}-\left(\frac{\pi}{4}\right)^6$

23. $\frac{\pi^3}{24}+\frac{1}{2}-\frac{\pi}{2}$

24. $\frac{\pi^3}{24}+\frac{\pi}{4}-1$

25. 2

26. $\frac{8}{15}\left(1+x^{\frac{3}{2}}\right)^{\frac{5}{4}}+c$

27. $\frac{2}{5}x^{\frac{5}{2}}+\frac{8}{3}x^{\frac{3}{2}}+8x^{\frac{1}{2}}+c$

28. $\frac{1}{3}(1+x^2)^{\frac{3}{2}}-(1+x^2)^{\frac{1}{2}}+c$

29. $-\frac{2\sqrt{1-x^3}}{3}+c$

30. $\frac{x}{2}+\frac{\sin 6x}{12}+c$

31. $\frac{\sec^3(x^2+3)}{3}+c$

32. $-\frac{\operatorname{cosec}^4 x}{4}+c$

33. $-\frac{1}{3}\cot x^3+c$

34. $-\frac{1}{2}\left(\frac{\cos 7x}{7}+\cos x\right)+c$

35. $\frac{1}{2\pi}\cos\frac{\pi}{x^2}+c$

36. $\frac{3(3x^2+10x+2)^{\frac{1}{3}}}{2}+c$

37. $x\sec x-\ln|\sec x+\tan x|+c$

38. $\frac{\sin 2x}{4}-\frac{x\cos 2x}{2}+c$

39. $\frac{(x^2+3)^{\frac{5}{2}}}{5}-(x^2+3)^{\frac{3}{2}}+c$

40. $\frac{x^3\sin x^3}{3}+\frac{\cos x^3}{3}+c$

41. $\frac{\sin 6x}{3}+\frac{7x\sin 6x}{6}+\frac{7\cos 6x}{36}+c$

42. $2(\sqrt{3}-1)$

43. $\frac{\pi}{2}$

44. $\frac{\pi^2}{4}-2$

45. $\frac{7^6-3^6}{24}$

46. 0

47. $2\sqrt{2}-2$

48. $2-\sqrt{2}$

49. $\frac{5}{4}$

50. $\frac{50}{3}$

51. $\frac{1}{4}$

52. $1$

53. $1-\frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}$

54. $\frac{\sqrt{3}}{2} : \frac{\sqrt{3}}{4}$

55. (*i*) $\frac{11}{6}$ (*ii*) $0$

56. $\frac{4}{x^2-1}$

57. $\frac{x^3}{x^2+9}-\frac{x^2}{x+4}+x\ln(x^2+9)-2x\ln(x+4)$

58. $\frac{1-2\ln(2x)}{x^3}$

59. $\frac{2}{x}+2x\cot x^2-\frac{1}{x}\sin(\ln x)$

60. $\frac{1}{x\ln x}$

61. $e^{\sin x}[(1+x^2)\cos x+2x]$

$$+e^{-x\cos x}(x\sin x-\cos x)+\frac{1}{x}$$

62. $\sec x$

63. $\operatorname{cosec} x$

64. $-\dfrac{\frac{y}{x}+\ln y}{\frac{x}{y}+\ln x}$

65. $\dfrac{-(3+\ln x+e^{y}\cos x)}{e^{y}\sin x}$

66. $\dfrac{y\sin x-y\sec^2(xy)-\frac{e^{y}}{x}}{e^{y}\ln x+\cos x+x\sec^2(xy)}$

67. $-\dfrac{x\,e^{2y}+y^2\,e^{2x}}{x^2e^{2y}+ye^{2x}}$

68. $\dfrac{y[\sin(xy)-e^{xy}]-\cot x}{x[e^{xy}-\sin(xy)]}$

69. $\frac{x}{2}\Big[\sin(\ln x)-\cos(\ln x)\Big]+c$

70. $\frac{2}{3}x^{\frac{3}{2}}\left\{\ln(3x)-\frac{2}{3}\right\}+c$

71. $\frac{\cos 2x}{2}\Big[1-\ln(\cos 2x)\Big]+c$

72. $\frac{e^{2x}}{4}(2x^2-2x+1)+c$

73. $\ln|x+\cos x|+c$

74. $\ln|\ln x|+c$

75. $-\sqrt{a^2-x^2}+c$

76. $\frac{3}{2}\ln\left|1+x^{\frac{4}{3}}\right|+c$

77. $-4 \ln | 1+\cot x | +c$

78. $-\frac{1}{2} e^{-\sec 2x}+c$

79. $\left(x-\frac{x^3}{3}\right) \ln x-x+\frac{x^3}{9}+c$

80. $-e^{\cos^2 x}+c$

81. $\frac{2}{3} (2+\ln x)^{\frac{3}{2}}+c$

82. $e^{-\frac{1}{x}}-\frac{1}{x}+c$

83. $-\frac{1}{4(1+e^{4x})}+c$

84. $\frac{1}{2}\left(e^{2x}-e^{-2x}+4x\right)+c$

85. $\frac{1}{3} \sin e^{x^3}+c$

86. $\frac{2}{5} \ln | x+2 | +\frac{3}{5}\ln | x-3 | +$

87. $\ln | x-3 | -\ln | x-2 | +c$

88. $-\ln | x | - \ln | x+2 |$
$+3 \ln | x-2 | +c$

89. $\frac{1}{6}\ln | x | +\frac{13}{10}\ln | x-2 |$
$-\frac{7}{15}\ln | x+3 | +c$

90. $x+\frac{1}{2}\left\{\ln \left|\frac{x-1}{x+1}\right| \right\}+c$

91. $x \ln x - x + c$

92. $\ln\left| 1+\sin x \right| +c$

93. $\frac{e^4}{2}+3e^2-\frac{7}{2}$

94. $x^2+3x-2 \ln | x+1 | +c$

95. $\frac{1}{8}\left[(1-e^{-2})^4+1\right]$

96. $\frac{1}{6}\left[(31+3e^3)^2-(5+3e)^2\right]$

97. $\frac{1}{5}\left(2-e^2-e^{-2}\right)$

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